Practical Projective Geometric Algebra

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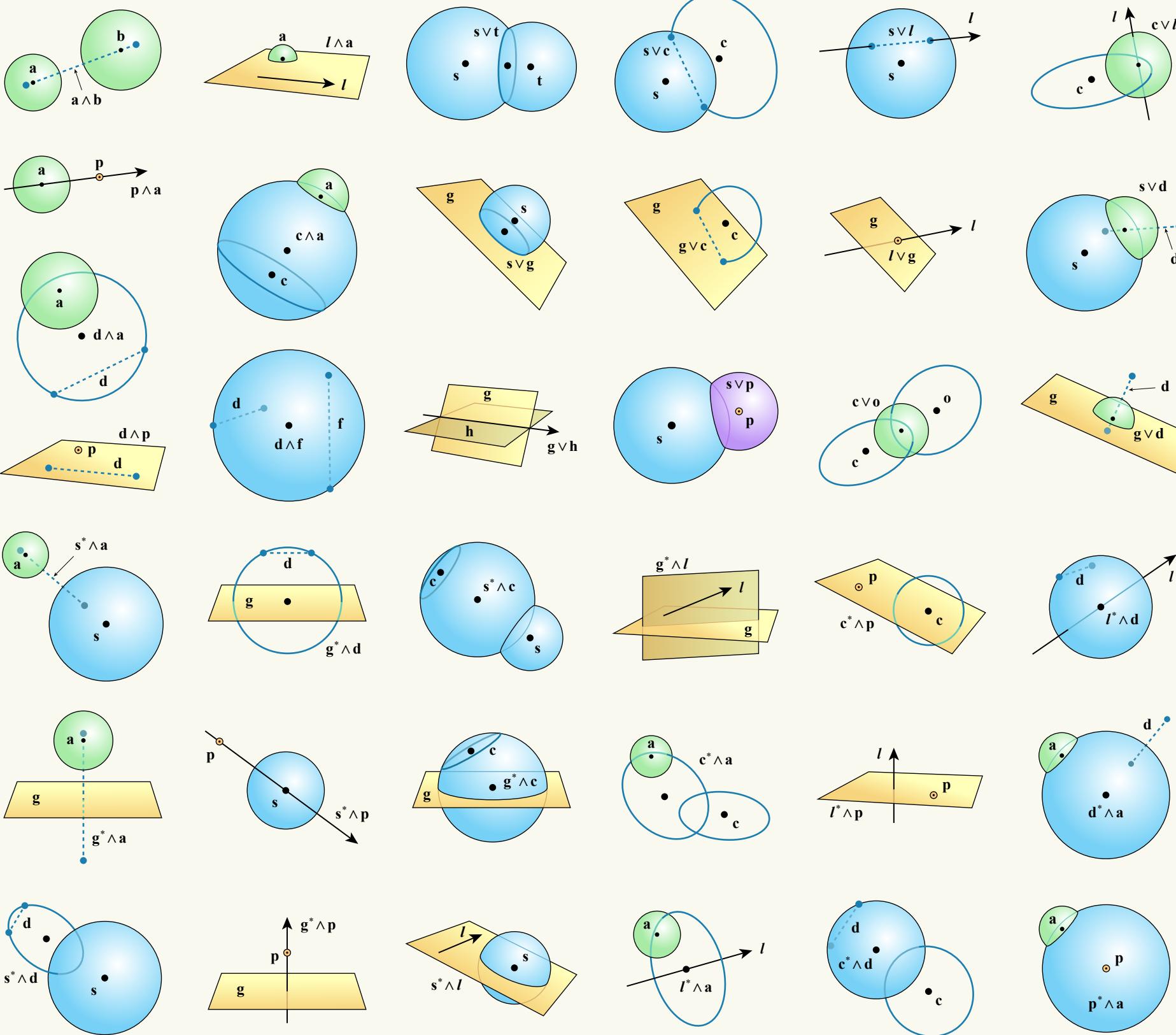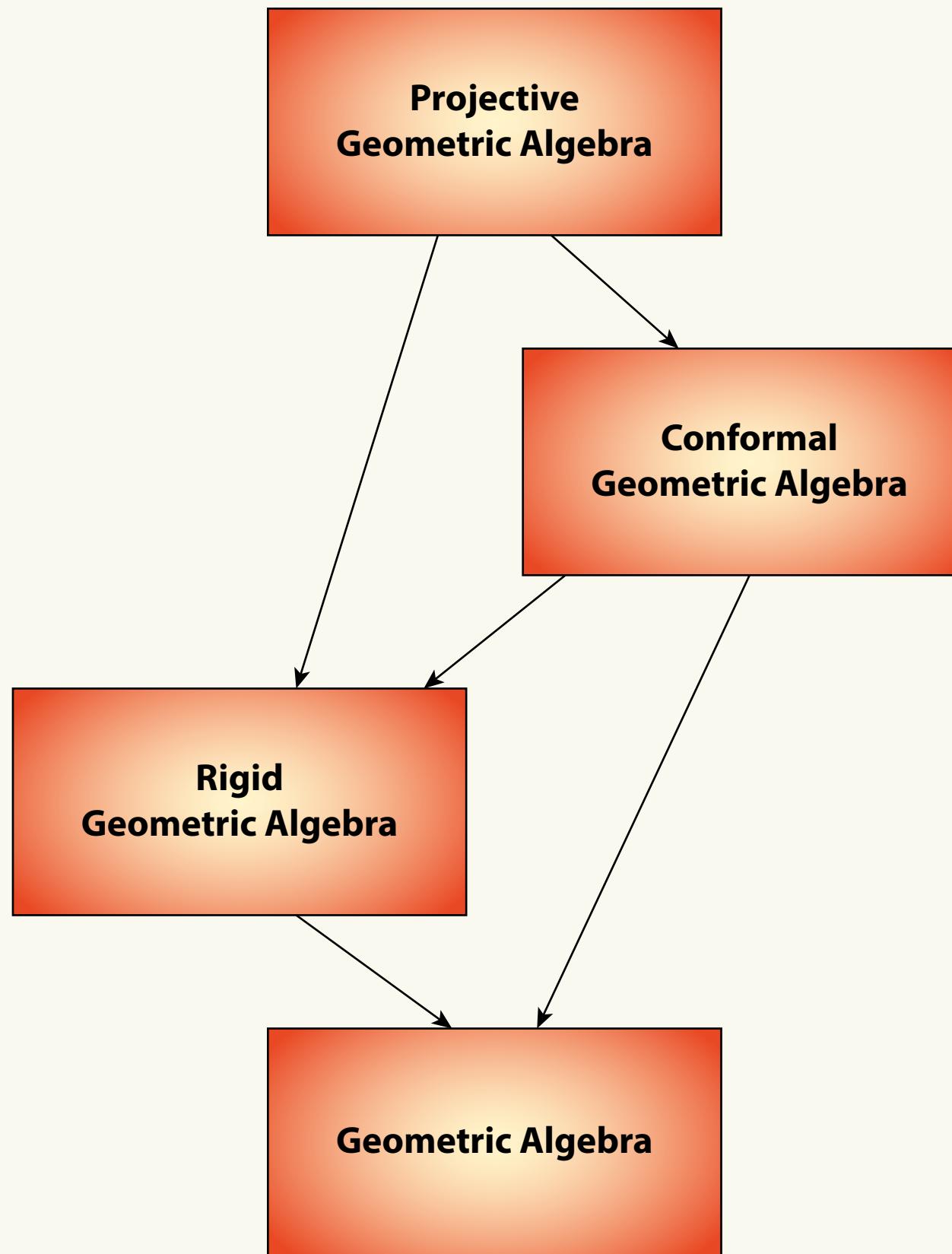
Algebras

Exterior / Grassmann Algebra

Wedge product

- Combines dimensions that are *present*
- Add *grades* of operands
- Repeated vectors give zero:

$$\mathbf{a} \wedge \mathbf{a} = 0$$

- Antisymmetric on vectors:

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$$

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Exterior / Grassmann Algebra

Duality means every product has an antiproduct

Antiwedge product $\vee$

- Combines dimensions that are *absent*
- Adds *antigrades* of operands

Geometric Algebra

Geometric product

- Includes wedge product
- Plus more information
- For vectors:

$$\mathbf{a} \wedge \mathbf{a} = \mathbf{a} \cdot \mathbf{a}$$

- Duality means there is a geometric antiproduct $\vee$

Exterior / Geometric Algebra

Wedge and antiwedge products perform geometric manipulation

- Join objects into higher-dimensional objects

- Intersect objects at lower-dimensional objects

- Project one object onto another object

Geometric products and antiproducts perform transformations

- Rotations, translations, reflections, inversions

- Dilations (scales), conformal transformations

Bivectors

Trivectors

$a \wedge b \wedge c$

$b \wedge c \wedge a$

$c \wedge a \wedge b$

$b \wedge a \wedge c$

$c \wedge b \wedge a$

$a \wedge c \wedge b$
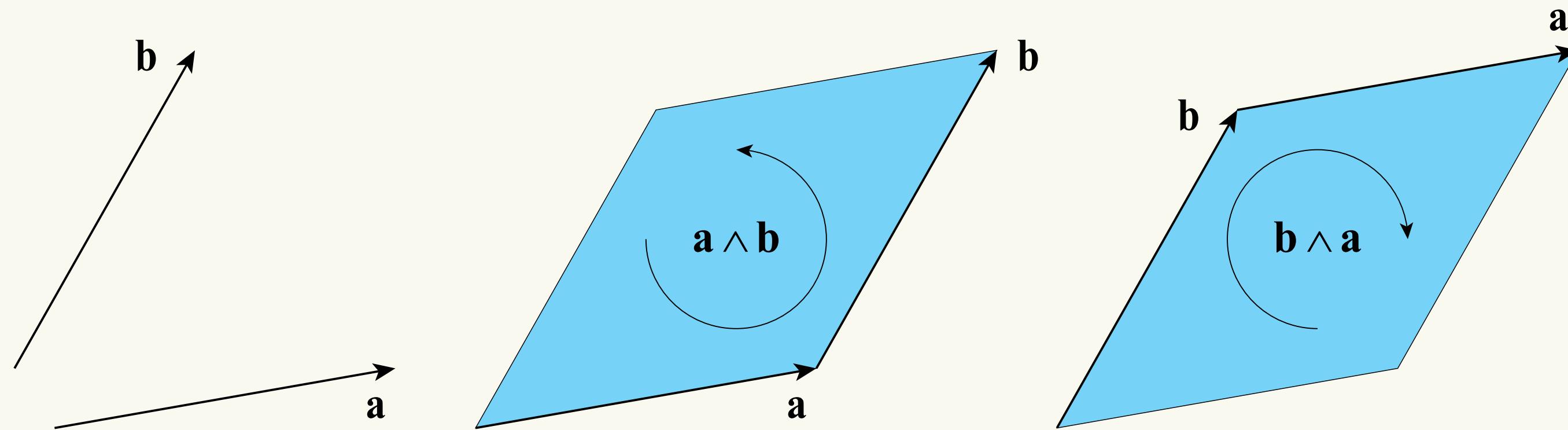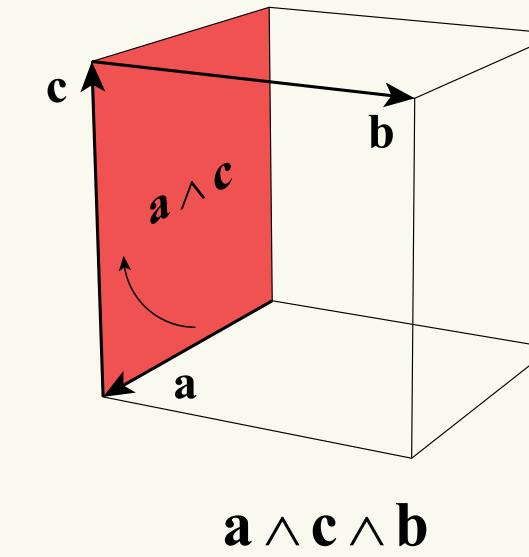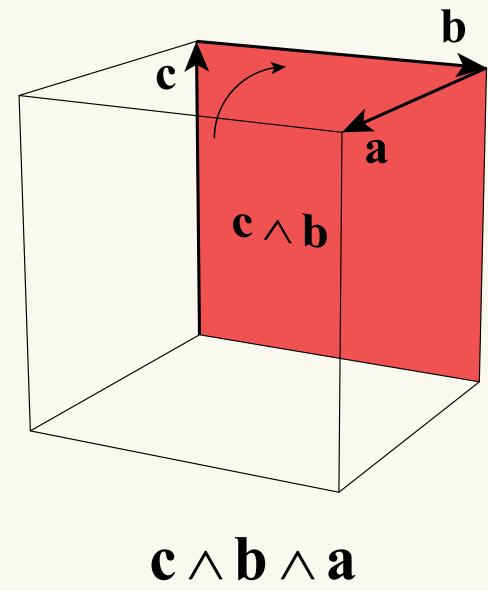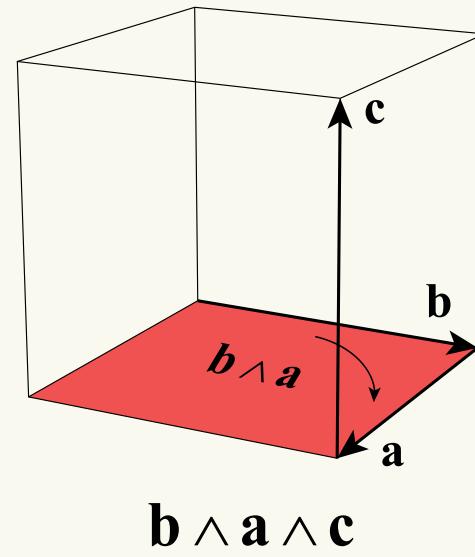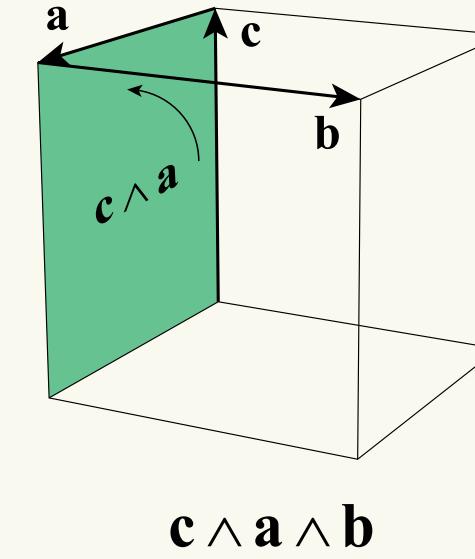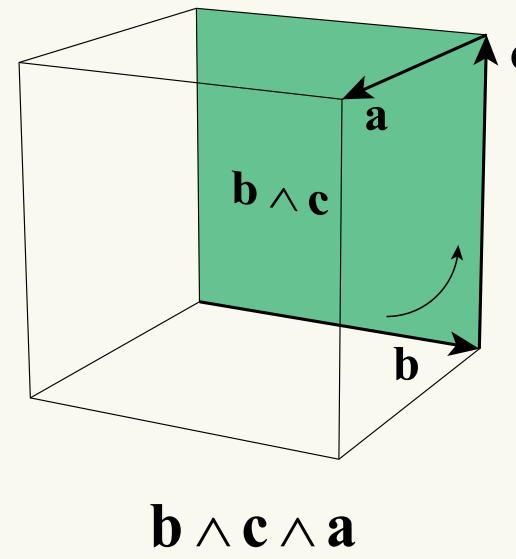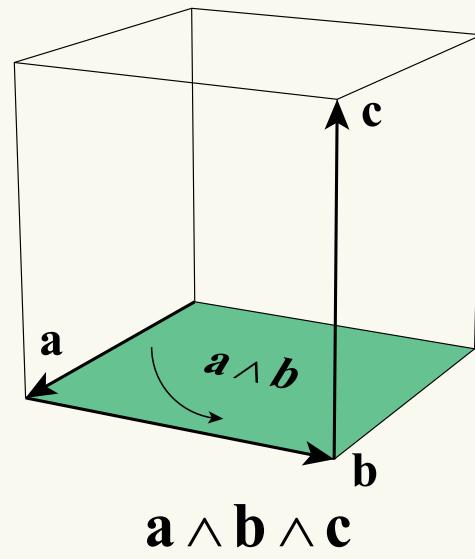
Pascal's Triangle
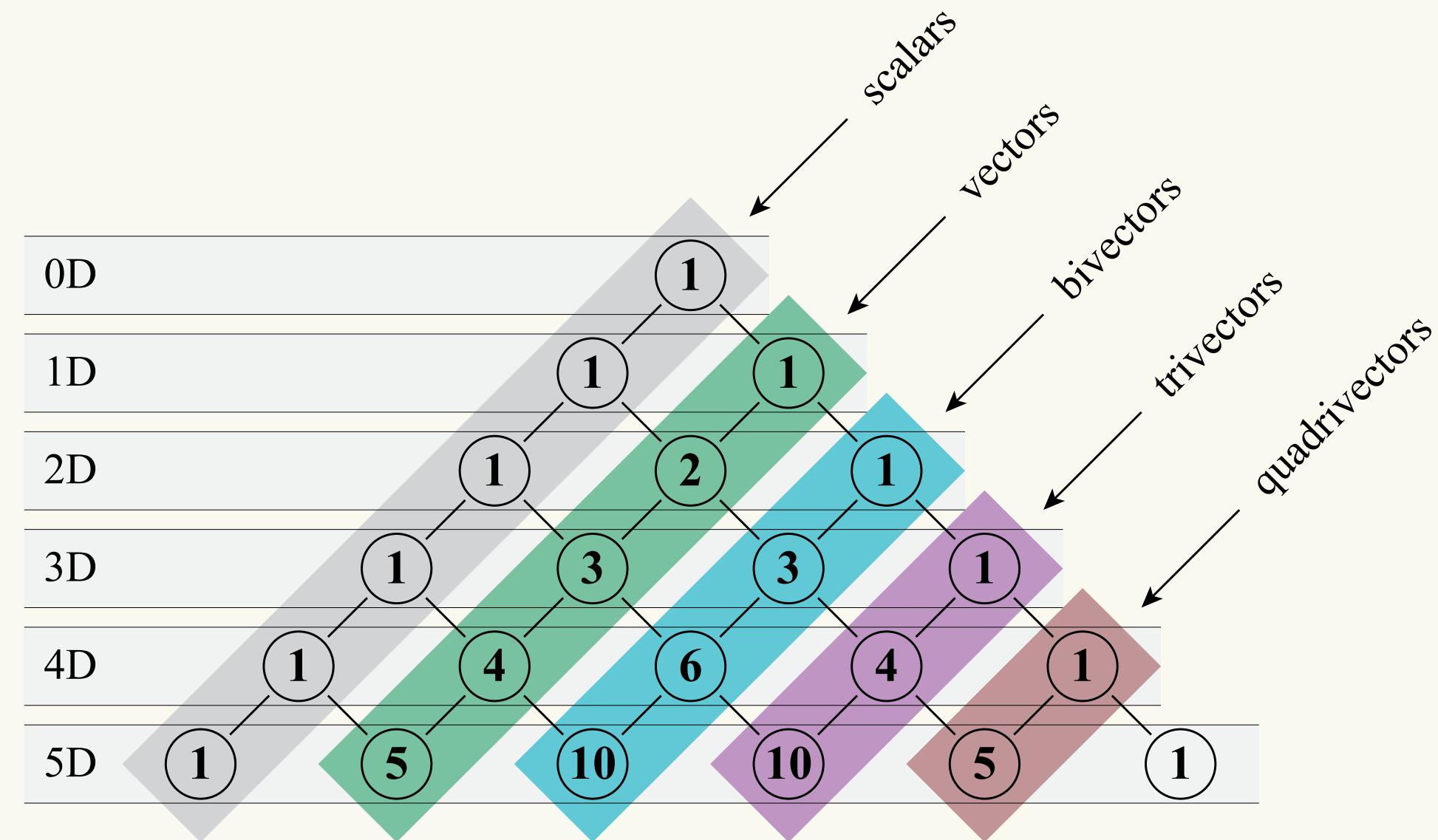
Rigid Geometric Algebra

Projective algebra with one extra dimension

Contains points, lines, planes

Can perform rotations, translations, screw transformations

Basis Elements

Type	Values	Grade / Antigrade	
Scalar	$\mathbf{1}$	0 / 4	
Vectors	$\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ $\mathbf{e}_4$	1 / 3	
Bivectors	$\mathbf{e}_{23} = \mathbf{e}_2 \wedge \mathbf{e}_3$ $\mathbf{e}_{31} = \mathbf{e}_3 \wedge \mathbf{e}_1$ $\mathbf{e}_{12} = \mathbf{e}_1 \wedge \mathbf{e}_2$ $\mathbf{e}_{43} = \mathbf{e}_4 \wedge \mathbf{e}_3$ $\mathbf{e}_{42} = \mathbf{e}_4 \wedge \mathbf{e}_2$ $\mathbf{e}_{41} = \mathbf{e}_4 \wedge \mathbf{e}_1$	2 / 2	
Trivectors / Antivectors	$\mathbf{e}_{321} = \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_1$ $\mathbf{e}_{412} = \mathbf{e}_4 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2$ $\mathbf{e}_{431} = \mathbf{e}_4 \wedge \mathbf{e}_3 \wedge \mathbf{e}_1$ $\mathbf{e}_{423} = \mathbf{e}_4 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$	3 / 1	
Antiscalar	$\mathbb{1} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$	4 / 0	
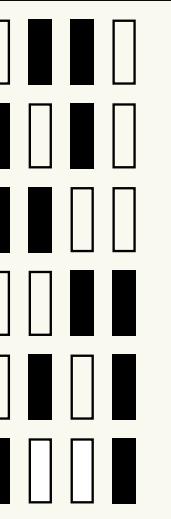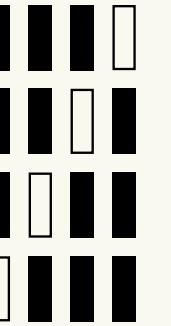
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Complements

Complement inverts dimensions

For basis elements $\mathbf{a}$, $\mathbf{a} \wedge \overline{\mathbf{a}} = \mathbb{1}$ $\underline{\mathbf{a}} \wedge \mathbf{a} = \mathbb{1}$

Basis element $\mathbf{a}$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$\mathbf{e}_{321}$	$\mathbf{e}_{412}$	$\mathbf{e}_{431}$	$\mathbf{e}_{423}$	$\mathbb{1}$
Right complement $\overline{\mathbf{a}}$	$\mathbb{1}$	$\mathbf{e}_{423}$	$\mathbf{e}_{431}$	$\mathbf{e}_{412}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_4$	$-\mathbf{e}_3$	$-\mathbf{e}_2$	$-\mathbf{e}_1$	1
Left complement $\underline{\mathbf{a}}$	$\mathbb{1}$	$-\mathbf{e}_{423}$	$-\mathbf{e}_{431}$	$-\mathbf{e}_{412}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$\mathbf{e}_4$	$\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_1$	1

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Homogeneous Coordinates

$$\mathbf{p} = (\, p_x \,,\, p_y \,,\, p_z \,,\, p_w \,)$$

w

$w = 1$

$\mathbf{p}/p_w$

y

x
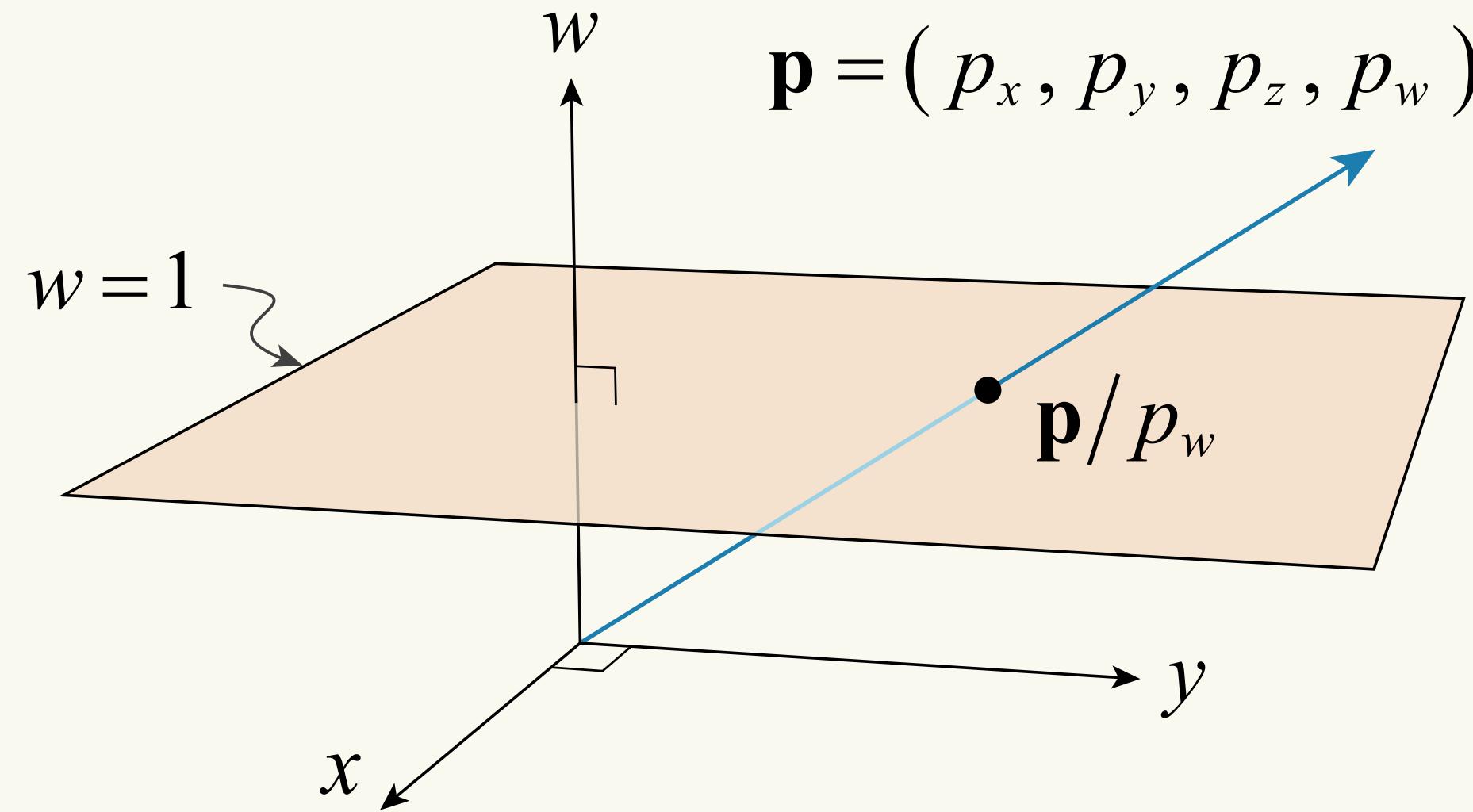
Plücker Coordinates
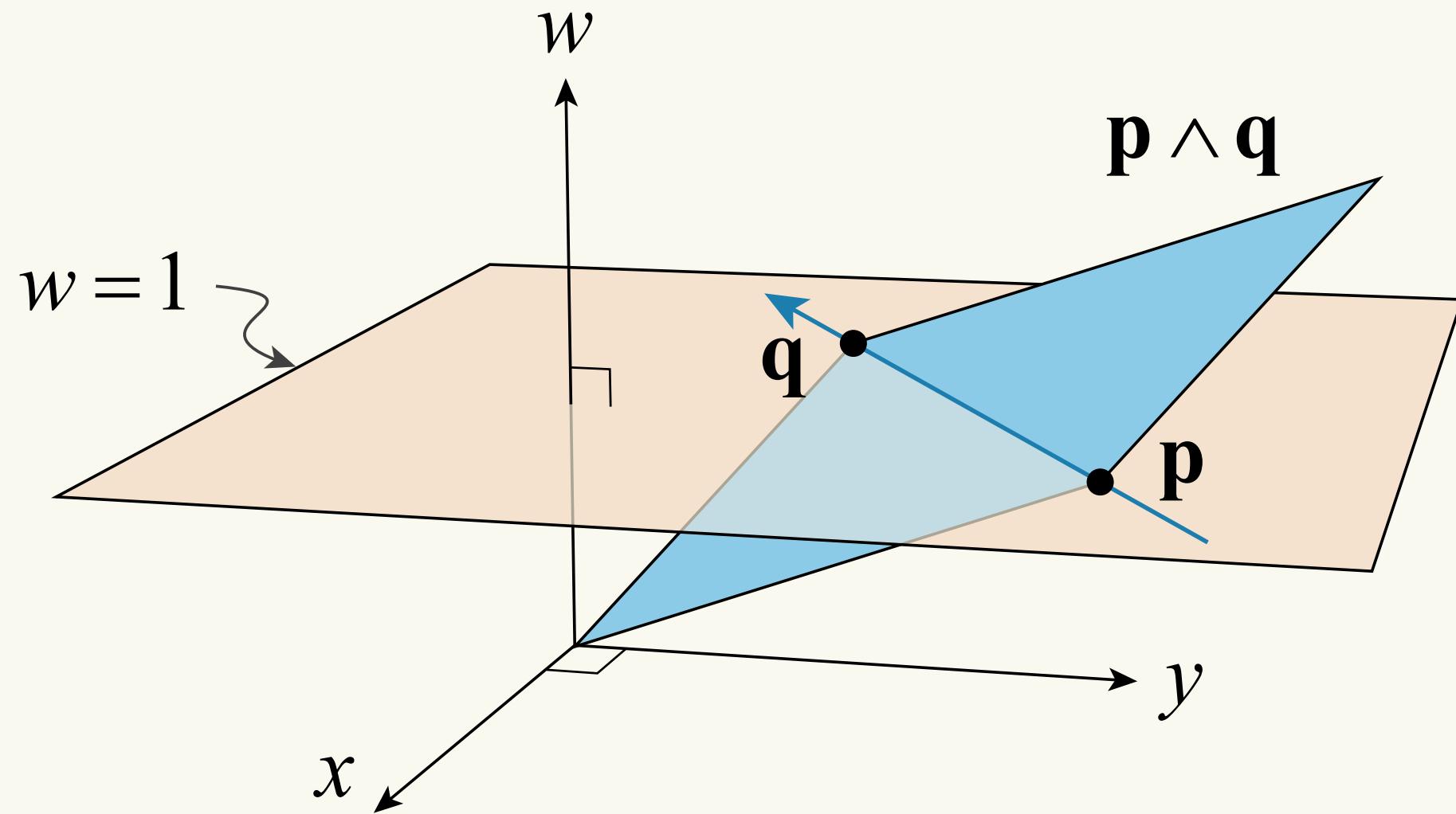
Planes
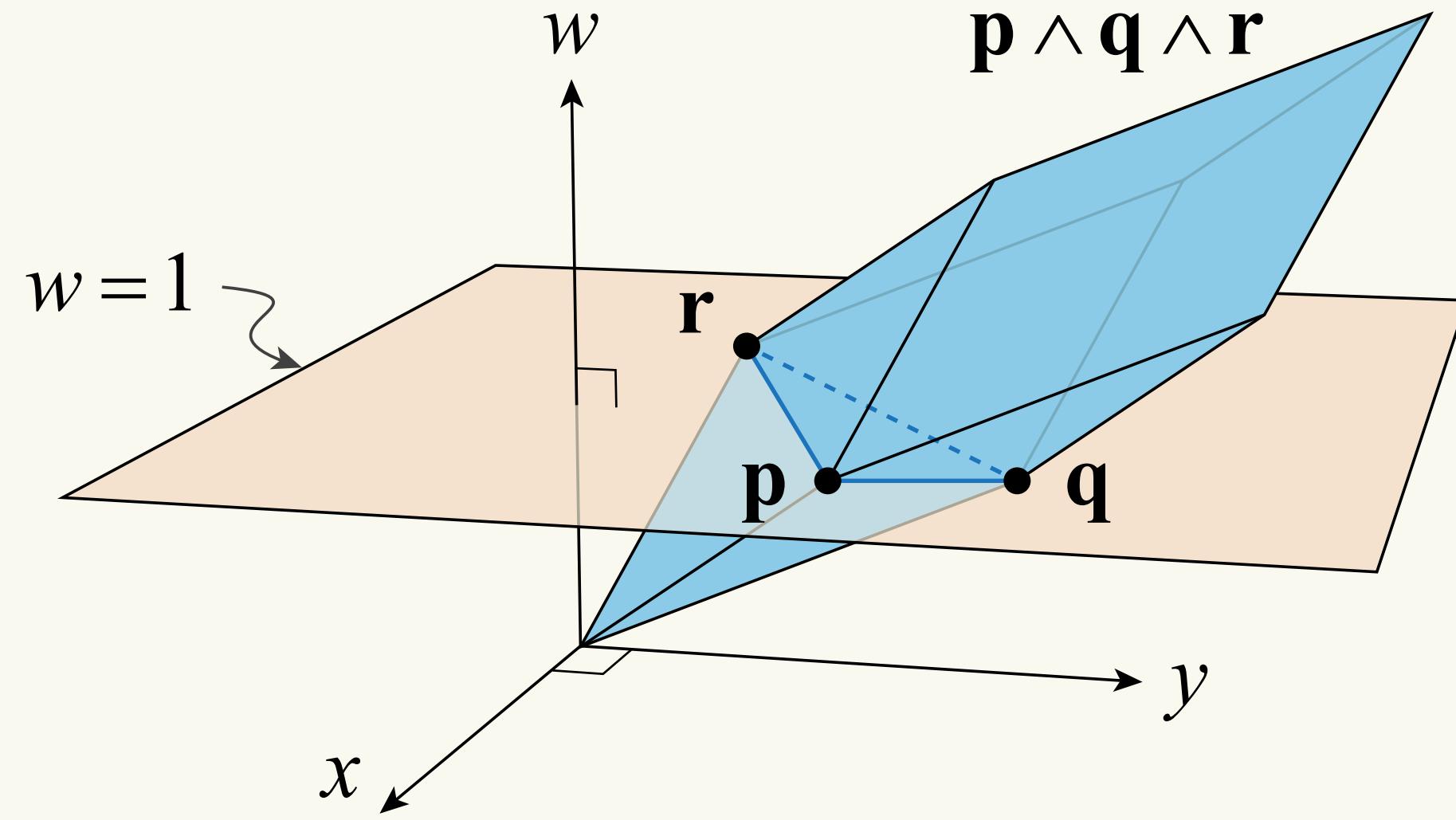
Bulk and Weight

Notation	Definition
$\mathbf{a}_\bullet$	Bulk of element $\mathbf{a}$. All components without factor of $\mathbf{e}_4$. $\mathbf{1}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{23}, \mathbf{e}_{31}, \mathbf{e}_{12}, \mathbf{e}_{321}$
$\mathbf{a}_\circ$	Weight of element $\mathbf{a}$. All components with factor of $\mathbf{e}_4$. $\mathbf{e}_4, \mathbf{e}_{41}, \mathbf{e}_{42}, \mathbf{e}_{43}, \mathbf{e}_{423}, \mathbf{e}_{431}, \mathbf{e}_{412}, \mathbb{1}$

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Bulk and Weight

Bulk contains positional information

- Distance from origin

Weight contains directional information

- Line direction

- Plane normal

Point

A 4D vector is a point

- Grade 1

$$\mathbf{p} = \boxed{p_x\,\mathbf{e}_1 + p_y\,\mathbf{e}_2 + p_z\,\mathbf{e}_3} + \boxed{p_w\,\mathbf{e}_4}$$

$$\mathbf{p}_\bullet = p_x\,\mathbf{e}_1 + p_y\,\mathbf{e}_2 + p_z\,\mathbf{e}_3$$

$$\mathbf{p}_\circ = p_w\,\mathbf{e}_4$$

The Origin

The origin is a point with no bulk

- Homogeneous coordinates $(0, 0, 0, w)$

$$\mathbf{p} = p_w \mathbf{e}_4$$

Line

A 4D bivector is a line
- Grade 2

$$l = \colorbox{purple}{$l_{vx}\,\mathbf{e}_{41} + l_{vy}\,\mathbf{e}_{42} + l_{vz}\,\mathbf{e}_{43}$} + \colorbox{green}{$l_{mx}\,\mathbf{e}_{23} + l_{my}\,\mathbf{e}_{31} + l_{mz}\,\mathbf{e}_{12}$}$$

$$l_\bullet = l_{mx}\,\mathbf{e}_{23} + l_{my}\,\mathbf{e}_{31} + l_{mz}\,\mathbf{e}_{12}$$

$$l_\circ = l_{vx}\,\mathbf{e}_{41} + l_{vy}\,\mathbf{e}_{42} + l_{vz}\,\mathbf{e}_{43}$$

Plane

A 4D trivector is a plane

- Grade 3

$$\mathbf{g} = g_x \, \mathbf{e}_{423} + g_y \, \mathbf{e}_{431} + g_z \, \mathbf{e}_{412} + g_w \, \mathbf{e}_{321}$$

$$\mathbf{g}\bullet = g_w \, \mathbf{e}_{321}$$

$$\mathbf{g}\circ = g_x \, \mathbf{e}_{423} + g_y \, \mathbf{e}_{431} + g_z \, \mathbf{e}_{412}$$

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Points at Infinity

A point with zero weight is a point at infinity

- Can be interpreted as a direction vector

$$\mathbf{p} = p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3$$

Lines at Infinity

A line with zero weight is a line at infinity

- Contains all points at infinity in directions parallel to moment

$$\boldsymbol{l} = l_{mx}\,\mathbf{e}_{23} + l_{my}\,\mathbf{e}_{31} + l_{mz}\,\mathbf{e}_{12}$$

Lines at Infinity

$$\boldsymbol{l} = l_{mx}\,\mathbf{e}_{23} + l_{my}\,\mathbf{e}_{31} + l_{mz}\,\mathbf{e}_{12}$$

∞

∞

∞

∞

$\mathbf{m}$
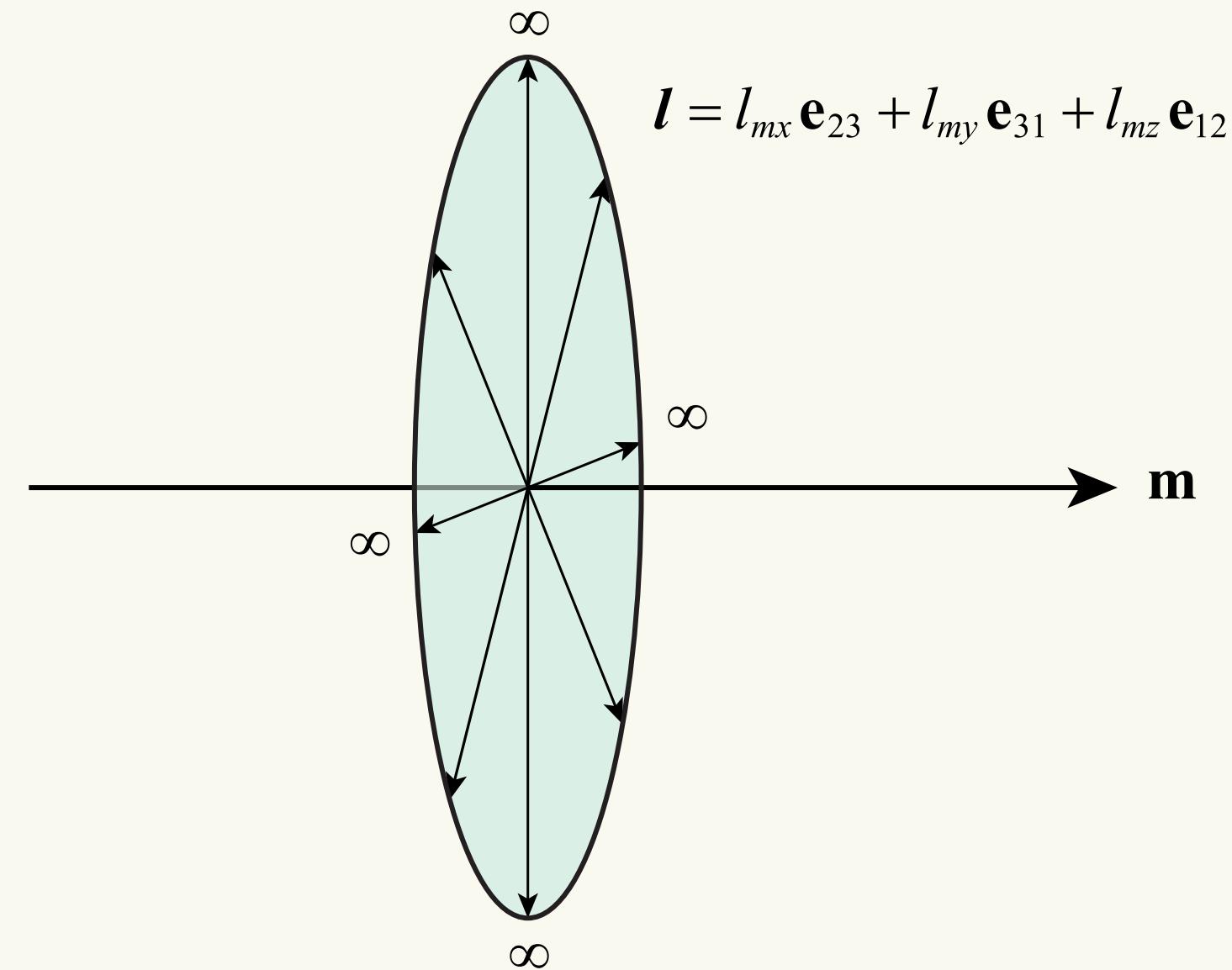
The Horizon

The plane with zero weight is called the *horizon*

- Contains all points at infinity

$$\mathbf{g} = g_w \mathbf{e}_{321}$$

The origin and horizon are duals of each other

Bulk and Weight

If the bulk is zero, then the object contains the origin

If the weight zero, then the horizon contains the object

Ratio bulk/weight is 3D magnitude, distance from origin

Norms

There are two dot products and two norms

Bulk norm $\qquad \|\mathbf{a}\|_{\bullet} = \sqrt{\mathbf{a} \cdot \tilde{\mathbf{a}}}$

Weight norm $\qquad \|\mathbf{a}\|_{\circ} = \sqrt{\mathbf{a} \circ \underset{\sim}{\mathbf{a}}}$

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Reverses

There are two reverse operations

Reverse $\tilde{\mathbf{a}}$, reverses multiplication order of vectors under wedge product

Antireverse $\underset{\sim}{\mathbf{a}}$, reverses multiplication order of antivectors under antiwedge product

Generalization of conjugates

Norms

Type	Bulk and Weight Norms	Projected Geometric Norm	Interpretation
Point $\mathbf{p}$	$\|\mathbf{p}\|_{\bullet} = \sqrt{p_x^2 + p_y^2 + p_z^2}$ $\|\mathbf{p}\|_{\circ} = \|p_w\|\mathbb{1}$	$\widehat{\|\mathbf{p}\|} = \dfrac{\sqrt{p_x^2 + p_y^2 + p_z^2}}{|p_w|}$	Distance from origin to point $\mathbf{p}$.
Line $\mathbf{L}$	$\|\mathbf{L}\|_{\bullet} = \sqrt{m_x^2 + m_y^2 + m_z^2}$ $\|\mathbf{L}\|_{\circ} = \mathbb{1}\sqrt{v_x^2 + v_y^2 + v_z^2}$	$\widehat{\|\mathbf{L}\|} = \sqrt{\dfrac{m_x^2 + m_y^2 + m_z^2}{v_x^2 + v_y^2 + v_z^2}}$	Perpendicular distance from origin to line $\mathbf{L}$.
Plane $\mathbf{f}$	$\|\mathbf{f}\|_{\bullet} = |f_w|$ $\|\mathbf{f}\|_{\circ} = \mathbb{1}\sqrt{f_x^2 + f_y^2 + f_z^2}$	$\widehat{\|\mathbf{f}\|} = \dfrac{|f_w|}{\sqrt{f_x^2 + f_y^2 + f_z^2}}$	Perpendicular distance from origin to plane $\mathbf{f}$.

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Duality

homogeneous point
$$\left(p_x, p_y, p_z, p_w \right)$$

homogeneous plane
$$\left(p_x, p_y, p_z, p_w \right)$$

$w = 1$

w

$1/r$ r

y

x

$$r = \sqrt{\left(\frac{p_x}{p_w}\right)^2 + \left(\frac{p_y}{p_w}\right)^2 + \left(\frac{p_z}{p_w}\right)^2}$$
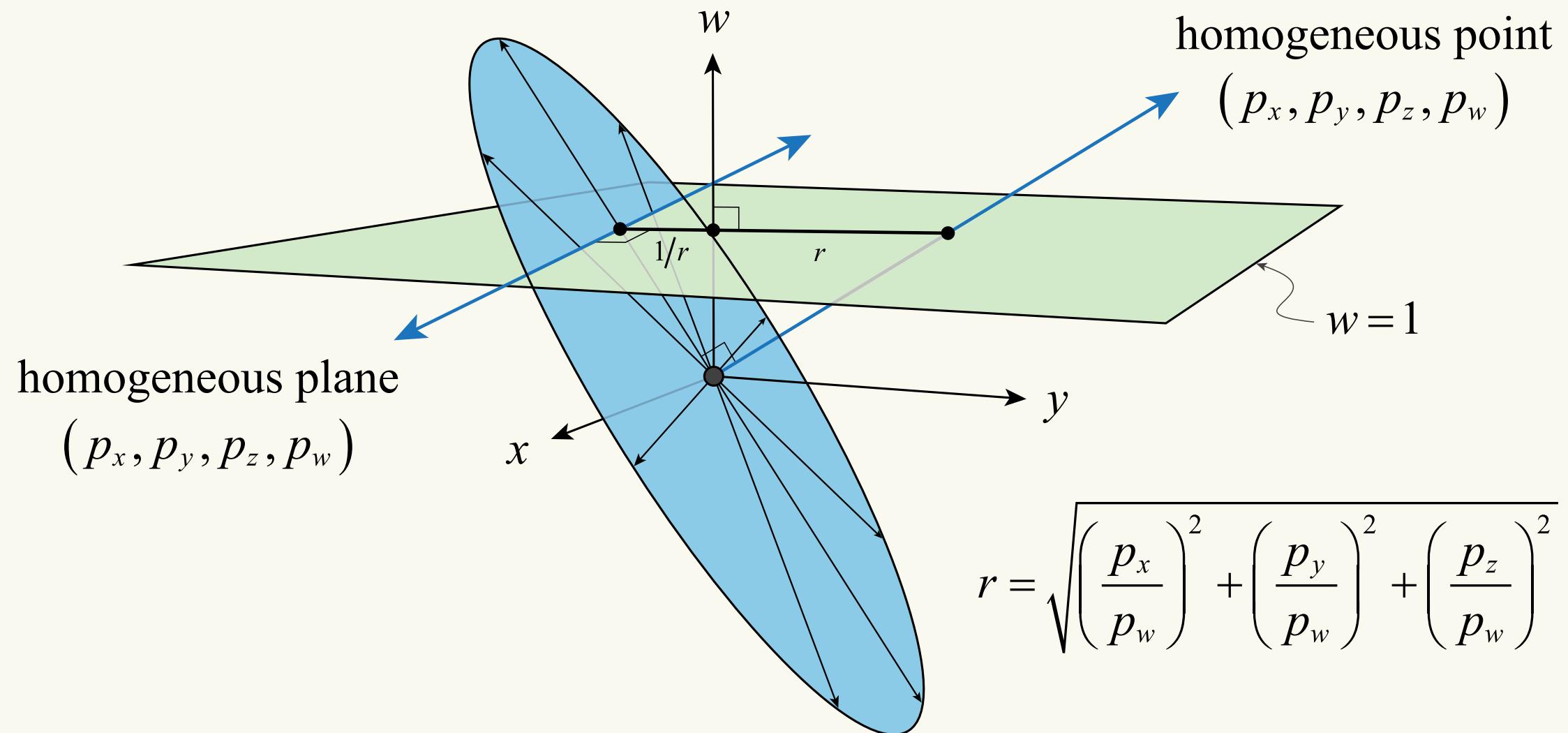
Duality

Every object is really two things at once
- One thing in *space*
- The dual of that thing in *antispace*

When any operation is performed, something happens in space, and something else happens in antispace

Join Operation

Formula	Illustration
Line containing points **p** and **q**. $\mathbf{p} \wedge \mathbf{q} = \left(p_w q_x - p_x q_w \right) \mathbf{e}_{41} + \left(p_y q_z - p_z q_y \right) \mathbf{e}_{23}$ $\qquad + \left(p_w q_y - p_y q_w \right) \mathbf{e}_{42} + \left(p_z q_x - p_x q_z \right) \mathbf{e}_{31}$ $\qquad + \left(p_w q_z - p_z q_w \right) \mathbf{e}_{43} + \left(p_x q_y - p_y q_x \right) \mathbf{e}_{12}$	
Plane containing line l and point **p**. $l \wedge \mathbf{p} = \left(l_{vy} p_z - l_{vz} p_y + l_{mx} p_w \right) \mathbf{e}_{423}$ $\qquad + \left(l_{vz} p_x - l_{vx} p_z + l_{my} p_w \right) \mathbf{e}_{431}$ $\qquad + \left(l_{vx} p_y - l_{vy} p_x + l_{mz} p_w \right) \mathbf{e}_{412}$ $\qquad - \left(l_{mx} p_x + l_{my} p_y + l_{mz} p_z \right) \mathbf{e}_{321}$	
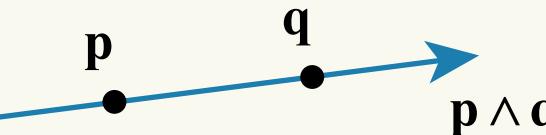
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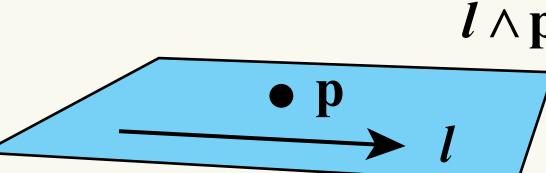
Edge Extrusion

The wedge product of a point and an edge's line is the plane extruded away from the point through the edge

- Portals
- Occluders
- Shadow regions
- ...

Meet Operation

Formula	Illustration
Line where planes **g** and **h** intersect. $\begin{aligned}\mathbf{g} \vee \mathbf{h} = &\left(g_z h_y - g_y h_z \right) \mathbf{e}_{41} + \left(g_x h_w - g_w h_x \right) \mathbf{e}_{23} \\ &+ \left(g_x h_z - g_z h_x \right) \mathbf{e}_{42} + \left(g_y h_w - g_w h_y \right) \mathbf{e}_{31} \\ &+ \left(g_y h_x - g_x h_y \right) \mathbf{e}_{43} + \left(g_z h_w - g_w h_z \right) \mathbf{e}_{12}\end{aligned}$	
Point where line l intersects plane **g**. $\begin{aligned}\mathbf{g} \vee l \ = &\left(g_z l_{my} - g_y l_{mz} + g_w l_{vx} \right) \mathbf{e}_1 \\ &+ \left(g_x l_{mz} - g_z l_{mx} + g_w l_{vy} \right) \mathbf{e}_2 \\ &+ \left(g_y l_{mx} - g_x l_{my} + g_w l_{vz} \right) \mathbf{e}_3 \\ &- \left(g_x l_{vx} + g_y l_{vy} + g_z l_{vz} \right) \mathbf{e}_4\end{aligned}$	
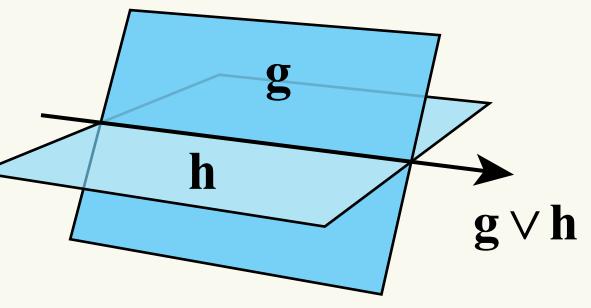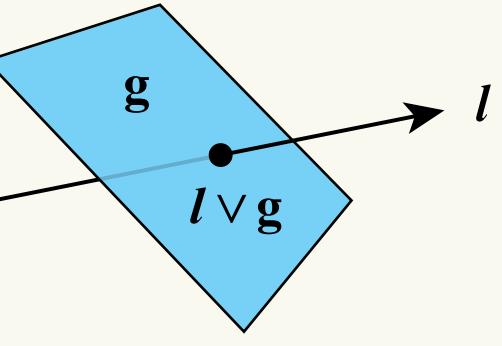
Line Crossing
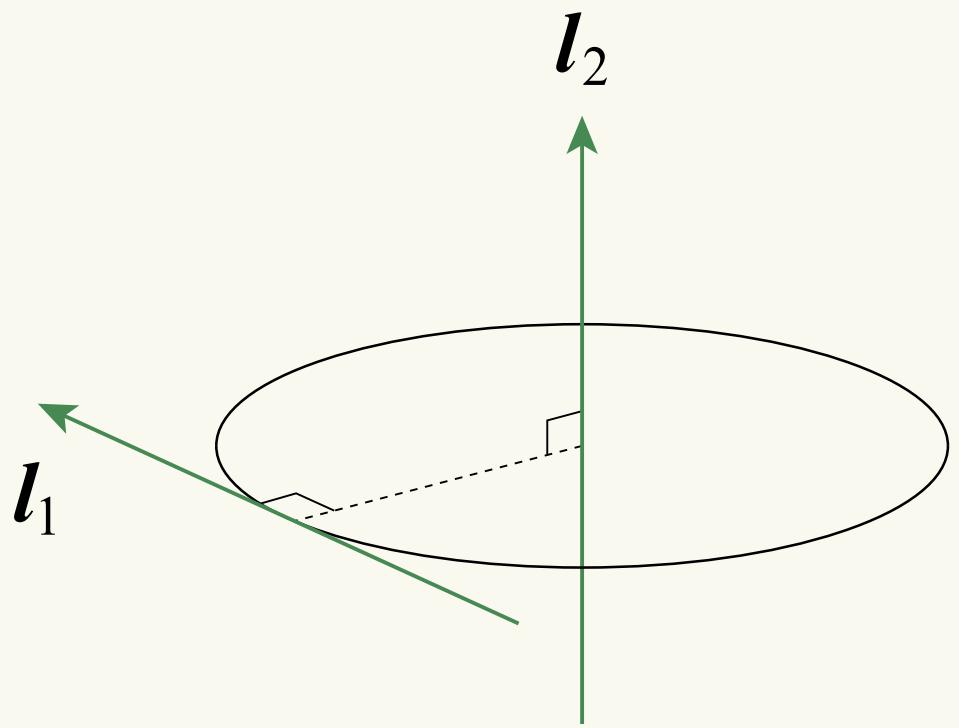
$$\boldsymbol{l}_1 \vee \boldsymbol{l}_2 > 0$$
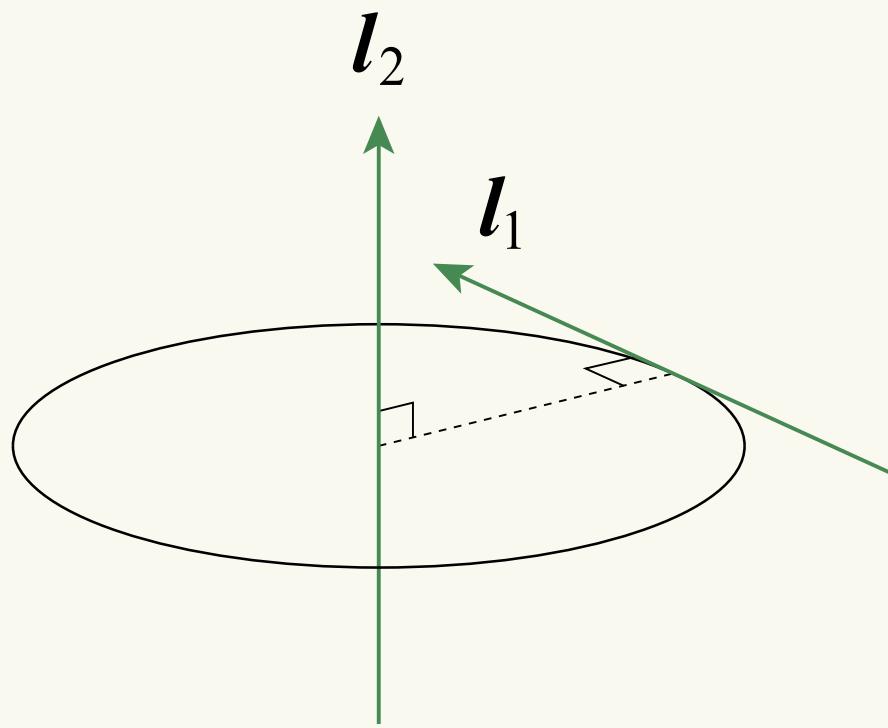
$$\boldsymbol{l}_1 \vee \boldsymbol{l}_2 < 0$$

Ray Passes Through Triangle

Check that ray passes on same side of all three edges

Translate so that one vertex is at origin to optimize

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Connect Operation

Connect constructs the object containing one object and orthogonal to another object

Uses weight complement of one object to extract directional information

Connect Operation

Formula	Illustration
Line perpendicular to plane **g** and containing point **p**. $\underline{\mathbf{g}_\circ} \wedge \mathbf{p} = (g_y p_z - g_z p_y)\,\mathbf{e}_{23}$ $\quad\quad + (g_z p_x - g_x p_z)\,\mathbf{e}_{31}$ $\quad\quad + (g_x p_y - g_y p_x)\,\mathbf{e}_{12}$ $\quad\quad - g_x p_w \mathbf{e}_{41} - g_y p_w \mathbf{e}_{42} - g_z p_w \mathbf{e}_{43}$	
Plane perpendicular to line l and containing point **p**. $\underline{l_\circ} \wedge \mathbf{p} = -l_{vx} p_w \mathbf{e}_{423} - l_{vy} p_w \mathbf{e}_{431} - l_{vz} p_w \mathbf{e}_{412}$ $\quad\quad + (l_{vx} p_x + l_{vy} p_y + l_{vz} p_z)\,\mathbf{e}_{321}$	
Plane perpendicular to plane **g** and containing line l. $\underline{\mathbf{g}_\circ} \wedge l = (g_z l_{vy} - g_y l_{vz})\,\mathbf{e}_{423}$ $\quad\quad + (g_x l_{vz} - g_z l_{vx})\,\mathbf{e}_{431}$ $\quad\quad + (g_y l_{vx} - g_x l_{vy})\,\mathbf{e}_{412}$ $\quad\quad - (g_x l_{mx} + g_y l_{my} + g_z l_{mz})\,\mathbf{e}_{321}$	
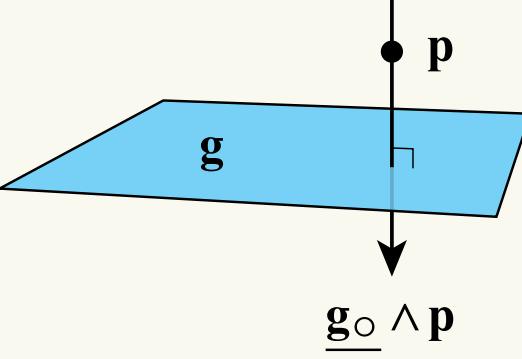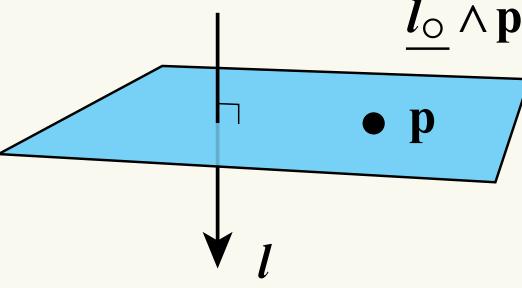
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Projections

Formula	Illustration
Projection of point $\mathbf{p}$ onto plane $\mathbf{g}$. $\left(\underline{\mathbf{g}_{\circ}} \wedge \mathbf{p}\right) \vee \mathbf{g} = \left(g_x^2 + g_y^2 + g_z^2\right)\mathbf{p}$ $\quad -\left(g_x p_x + g_y p_y + g_z p_z + g_w p_w\right)\left(g_x \mathbf{e}_1 + g_y \mathbf{e}_2 + g_z \mathbf{e}_3\right)$	
Projection of point $\mathbf{p}$ onto line l. $\left(\underline{l_{\circ}} \wedge \mathbf{p}\right) \vee l = \left(l_{vx} p_x + l_{vy} p_y + l_{vz} p_z\right)\mathbf{v}$ $\qquad + \left(l_{vy} l_{mz} - l_{vz} l_{my}\right) p_w \mathbf{e}_1$ $\qquad + \left(l_{vz} l_{mx} - l_{vx} l_{mz}\right) p_w \mathbf{e}_2$ $\qquad + \left(l_{vx} l_{my} - l_{vy} l_{mx}\right) p_w \mathbf{e}_3$ $\qquad + \left(l_{vx}^2 + l_{vy}^2 + l_{vz}^2\right) p_w \mathbf{e}_4$	
Projection of line l onto plane $\mathbf{g}$. $\left(\underline{\mathbf{g}_{\circ}} \wedge l\right) \vee \mathbf{g} = \left(g_x^2 + g_y^2 + g_z^2\right)\left(l_{vx} \mathbf{e}_{41} + l_{vy} \mathbf{e}_{42} + l_{vz} \mathbf{e}_{43}\right)$ $\quad - \left(l_{vx} g_x + l_{vy} g_y + l_{vz} g_z\right)\left(g_x \mathbf{e}_{41} + g_y \mathbf{e}_{42} + g_z \mathbf{e}_{43}\right)$ $\quad + \left(l_{mx} g_x + l_{my} g_y + l_{mz} g_z\right)\left(g_x \mathbf{e}_{23} + g_y \mathbf{e}_{31} + g_z \mathbf{e}_{12}\right)$ $\quad + \left(l_{vy} g_z - l_{vz} g_y\right) g_w \mathbf{e}_{23} + \left(l_{vz} g_x - l_{vx} g_z\right) g_w \mathbf{e}_{31}$ $\quad + \left(l_{vx} g_y - l_{vy} g_x\right) g_w \mathbf{e}_{12}$	
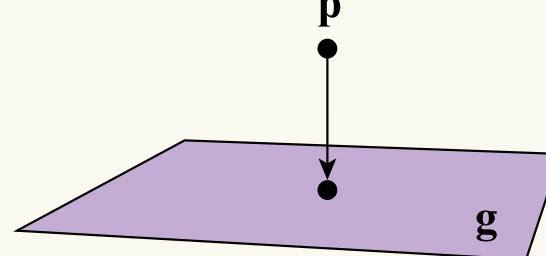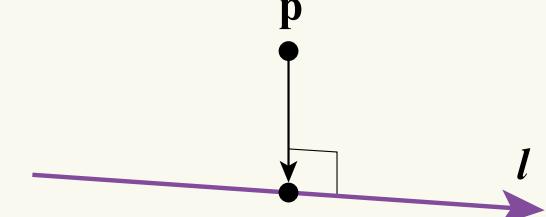
Anti-Projections

Formula	Illustration
Antiprojection of plane $\mathbf{g}$ onto point $\mathbf{p}$. $$\left(\underline{\mathbf{p}_\circ} \vee \mathbf{g} \right) \wedge \mathbf{p} = p_w^2 \left(g_x \mathbf{e}_{423} + g_y \mathbf{e}_{431} + g_z \mathbf{e}_{412} \right)$$ $$- \left(g_x p_x + g_y p_y + g_z p_z \right) p_w \mathbf{e}_{321}$$	
Antiprojection of line l onto point $\mathbf{p}$. $$\left(\underline{\mathbf{p}_\circ} \vee \mathit{l} \right) \wedge \mathbf{p} = p_w^2 \left(l_{vx} \mathbf{e}_{41} + l_{vy} \mathbf{e}_{42} + l_{vz} \mathbf{e}_{43} \right)$$ $$+ \left(l_{vz} p_y - l_{vy} p_z \right) p_w \mathbf{e}_{23}$$ $$+ \left(l_{vx} p_z - l_{vz} p_x \right) p_w \mathbf{e}_{31}$$ $$+ \left(l_{vy} p_x - l_{vx} p_y \right) p_w \mathbf{e}_{12}$$	
Antiprojection of plane $\mathbf{g}$ onto line l. $$\left(\underline{\mathit{l}_\circ} \vee \mathbf{g} \right) \wedge \mathit{l} = \left(l_{vx}^2 + l_{vy}^2 + l_{vz}^2 \right) \left(g_x \mathbf{e}_{423} + g_y \mathbf{e}_{431} + g_z \mathbf{e}_{412} \right)$$ $$- \left(l_{vx} g_x + l_{vy} g_y + l_{vz} g_z \right) \left(l_{vx} \mathbf{e}_{423} + l_{vy} \mathbf{e}_{431} + l_{vz} \mathbf{e}_{412} \right)$$ $$+ g_x \left(l_{vz} l_{my} - l_{vy} l_{mz} \right) \mathbf{e}_{321}$$ $$+ g_y \left(l_{vx} l_{mz} - l_{vz} l_{mx} \right) \mathbf{e}_{321}$$ $$+ g_z \left(l_{vy} l_{mx} - l_{vx} l_{my} \right) \mathbf{e}_{321}$$	
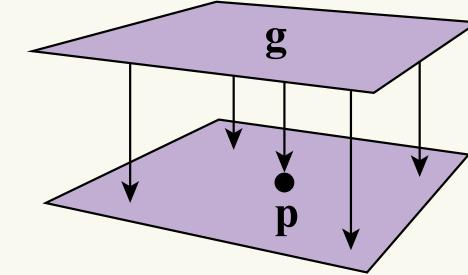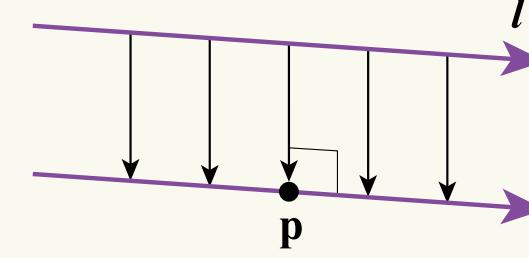
GDC
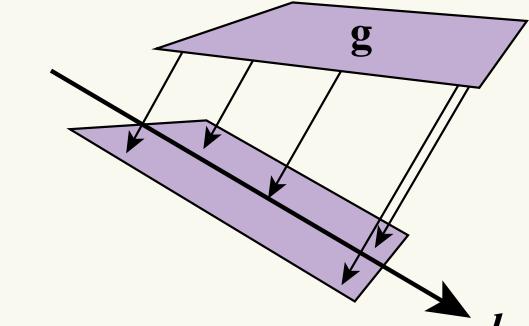
Euclidean Distances

Formula	Interpretation
$\sqrt{\left(q_x p_w - p_x q_w\right)^2 + \left(q_y p_w - p_y q_w\right)^2 + \left(q_z p_w - p_z q_w\right)^2} + \vert\, p_w q_w \,\vert\, \mathbb{1}$	Distance d between points $\mathbf{p}$ and $\mathbf{q}$.
$\sqrt{\left(l_{vy} p_z - l_{vz} p_y + l_{mx} p_w\right)^2 + \left(l_{vz} p_x - l_{vx} p_z + l_{my} p_w\right)^2 + \left(l_{vx} p_y - l_{vy} p_x + l_{mz} p_w\right)^2}$ $+\, \mathbb{1}\sqrt{p_w^2 \left(l_{vx}^2 + l_{vy}^2 + l_{vz}^2\right)}$	Perpendicular distance d between point $\mathbf{p}$ and line l.
$\vert\, p_x g_x + p_y g_y + p_z g_z + p_w g_w \,\vert + \mathbb{1}\sqrt{p_w^2 \left(g_x^2 + g_y^2 + g_z^2\right)}$	Perpendicular distance d between point $\mathbf{p}$ and plane $\mathbf{g}$.
$\vert\, l_{vx} k_{mx} + l_{vy} k_{my} + l_{vz} k_{mz} + k_{vx} l_{mx} + k_{vy} l_{my} + k_{vz} l_{mz} \,\vert$ $+\, \mathbb{1}\sqrt{\left(l_{vy} k_{vz} - l_{vz} k_{vy}\right)^2 + \left(l_{vz} k_{vx} - l_{vx} k_{vz}\right)^2 + \left(l_{vx} k_{vy} - l_{vy} k_{vx}\right)^2}$	Perpendicular distance d between skew lines $\mathbf{k}$ and l.

GDC

Geometric Product

Traditionally implied by juxtaposition

However, there are two products, just like wedge/antiwedge

We use notation $\wedge$ and $\vee$ for geometric product
and geometric antiproduct

"Wedge-dot" and "Antiwedge-dot"

Geometric Product

For geometric products, we define a *metric*:

$$\mathbf{e}_1 \wedge \mathbf{e}_1 = 1 \qquad\qquad \mathbf{e}_1 \vee \mathbf{e}_1 = \mathbb{1}$$

$$\mathbf{e}_2 \wedge \mathbf{e}_2 = 1 \qquad\qquad \mathbf{e}_2 \vee \mathbf{e}_2 = \mathbb{1}$$

$$\mathbf{e}_3 \wedge \mathbf{e}_3 = 1 \qquad\qquad \mathbf{e}_3 \vee \mathbf{e}_3 = \mathbb{1}$$

$$\mathbf{e}_4 \wedge \mathbf{e}_4 = 0 \qquad\qquad \mathbf{e}_4 \vee \mathbf{e}_4 = 0$$

Geometric Product

Sandwiches with geometric product or antiproduct perform transformations

Motor = MOtion operaTOR

Flector = reFLECtion operaTOR

Motor

General form of a motor

$$\mathbf{Q} = Q_{vx}\,\mathbf{e}_{41} + Q_{vy}\,\mathbf{e}_{42} + Q_{vz}\,\mathbf{e}_{43} + Q_{vw}\,\mathbb{1} + Q_{mx}\,\mathbf{e}_{23} + Q_{my}\,\mathbf{e}_{31} + Q_{mz}\,\mathbf{e}_{12} + Q_{mw}\,\mathbf{1}$$

Performs any combo of rotations and translations
- Proper Euclidean transformations
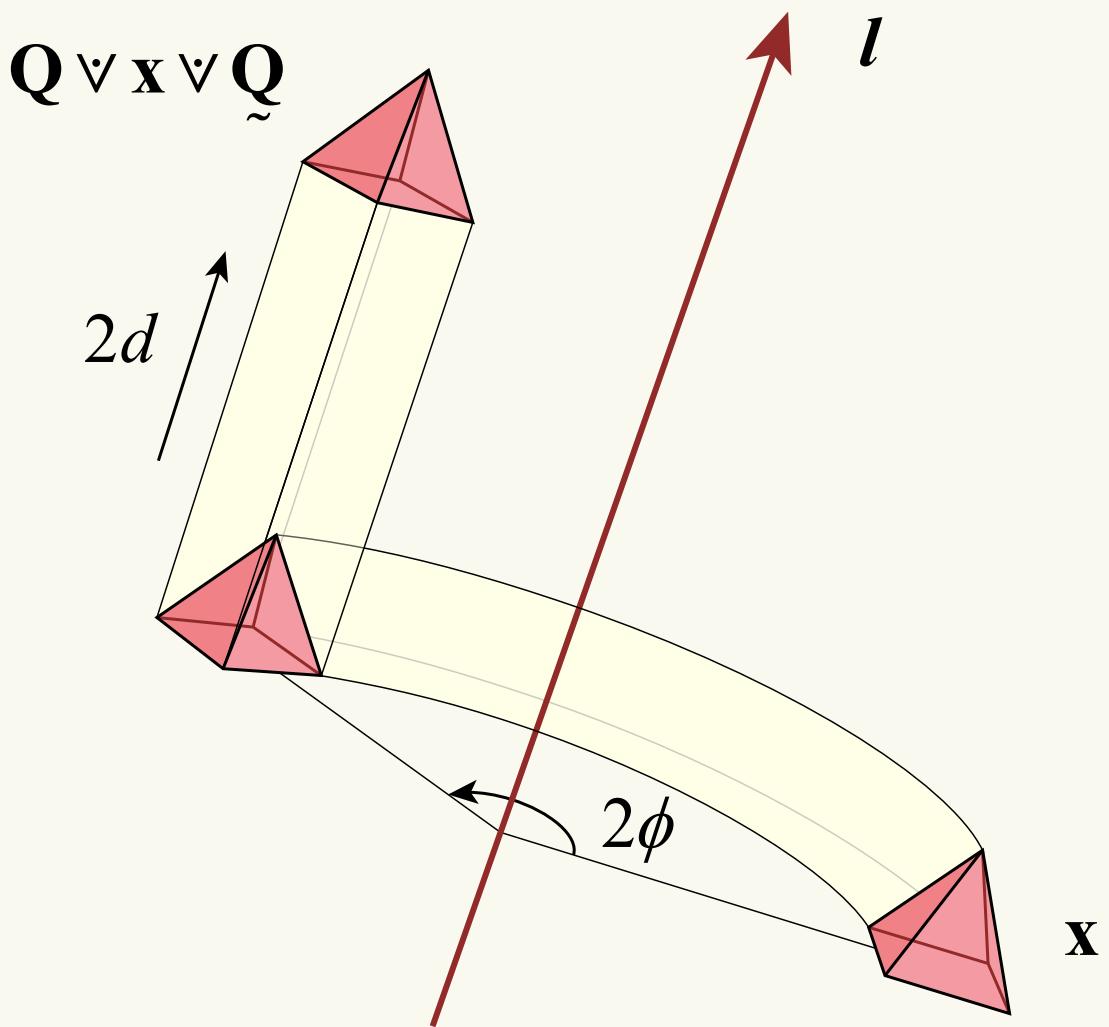
Motor

Flector

General form of a flector

$$\mathbf{F} = F_{px}\,\mathbf{e}_1 + F_{py}\,\mathbf{e}_2 + F_{pz}\,\mathbf{e}_3 + F_{pw}\,\mathbf{e}_4 + F_{gx}\,\mathbf{e}_{423} + F_{gy}\,\mathbf{e}_{431} + F_{gz}\,\mathbf{e}_{412} + F_{gw}\,\mathbf{e}_{321}$$

Performs any combo of rotations and translations plus an odd number of reflections

- Improper Euclidean transformations

Flector

$$\underline{\mathbf{g}}_\circ \wedge \mathbf{p}$$

$$2\phi$$

$$\mathbf{x}$$

$$\mathbf{p}$$

$$\mathbf{g}$$

$$-\mathbf{F} \veebar \mathbf{x} \veebar \underset{\sim}{\mathbf{F}}$$
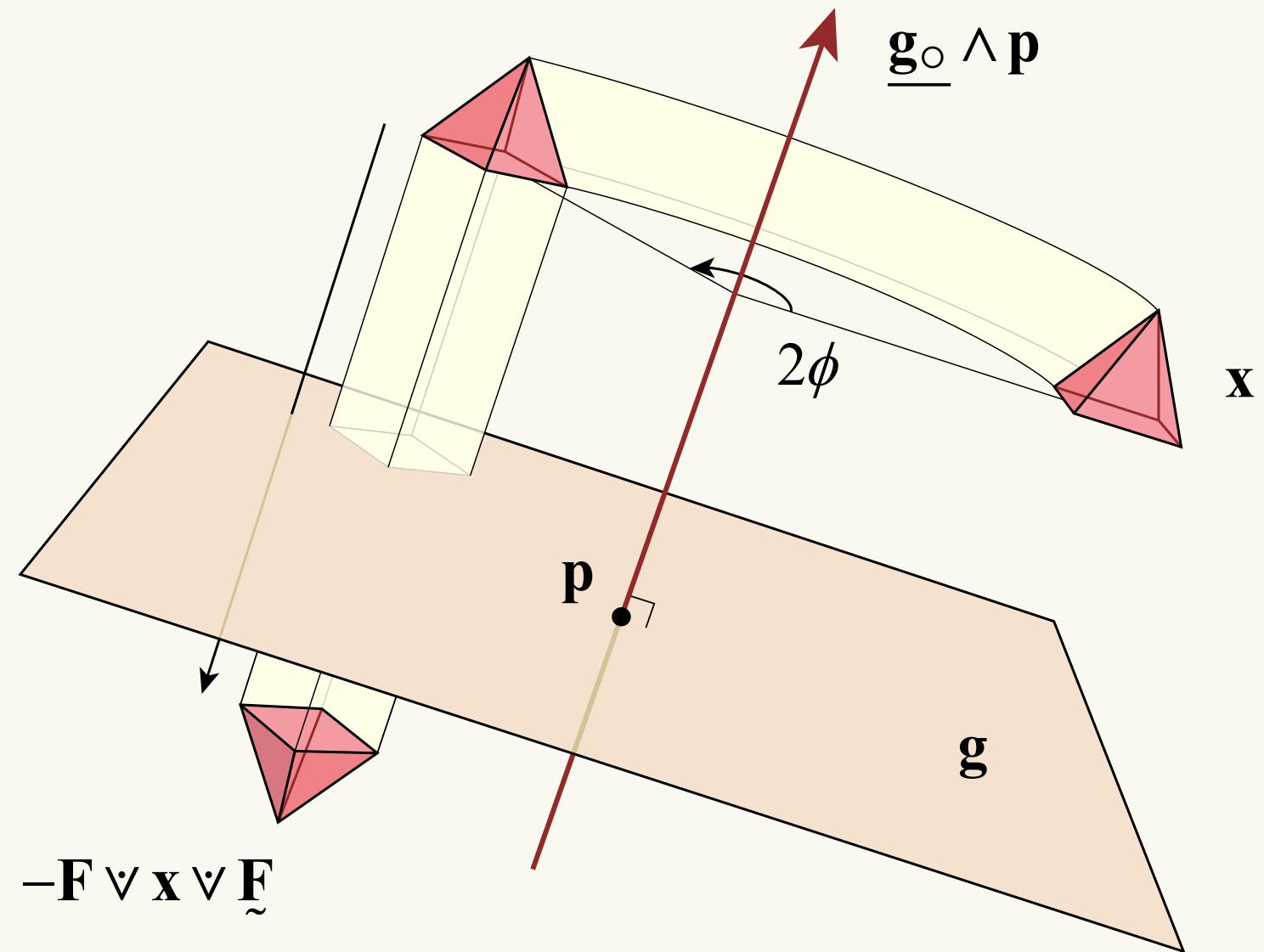
Motor Transforming Point

Can be reduced to 33 multiply-adds, or 30 mads if $p_w = 1$

$$\mathbf{Q} \veebar \mathbf{p} \veebar \underset{\sim}{\mathbf{Q}} = \left[(1 - 2Q_{vy}^2 - 2Q_{vz}^2)p_x + 2(Q_{vx}Q_{vy} - Q_{vz}Q_{vw})p_y + 2(Q_{vz}Q_{vx} + Q_{vy}Q_{vw})p_z + 2(Q_{vy}Q_{mz} - Q_{vz}Q_{my} + Q_{vw}Q_{mx} - Q_{vx}Q_{mw})p_w \right] \mathbf{e}_1$$

$$+ \left[(1 - 2Q_{vz}^2 - 2Q_{vx}^2))p_y + 2(Q_{vy}Q_{vz} - Q_{vx}Q_{vw})p_z + 2(Q_{vx}Q_{vy} + Q_{vz}Q_{vw})p_x + 2(Q_{vz}Q_{mx} - Q_{vx}Q_{mz} + Q_{vw}Q_{my} - Q_{vy}Q_{mw})p_w \right] \mathbf{e}_2$$

$$+ \left[(1 - 2Q_{vx}^2 - 2Q_{vy}^2))p_z + 2(Q_{vz}Q_{vx} - Q_{vy}Q_{vw})p_x + 2(Q_{vy}Q_{vz} + Q_{vx}Q_{vw})p_y + 2(Q_{vx}Q_{my} - Q_{vy}Q_{mx} + Q_{vw}Q_{mz} - Q_{vz}Q_{mw})p_w \right] \mathbf{e}_3$$

$$+ p_w \mathbf{e}_4$$

3x4 matrix transforming point only needs 12 mads,
or 9 mads if $p_w = 1$

Motor Transforming Line

Can be reduced to 63 multiply-adds

$$\mathbf{Q} \vee l \vee \tilde{\mathbf{Q}} = \left[(1 - 2Q_{vy}^2 - 2Q_{vz}^2)l_{vx} + 2(Q_{vx}Q_{vy} - Q_{vz}Q_{vw})l_{vy} + 2(Q_{vz}Q_{vx} + Q_{vy}Q_{vw})l_{vz} \right] \mathbf{e}_{41}$$

$$+ \left[(1 - 2Q_{vz}^2 - 2Q_{vx}^2)l_{vy} + 2(Q_{vy}Q_{vz} - Q_{vx}Q_{vw})l_{vz} + 2(Q_{vx}Q_{vy} + Q_{vz}Q_{vw})l_{vx} \right] \mathbf{e}_{42}$$

$$+ \left[(1 - 2Q_{vx}^2 - 2Q_{vy}^2)l_{vz} + 2(Q_{vz}Q_{vx} - Q_{vy}Q_{vw})l_{vx} + 2(Q_{vy}Q_{vz} + Q_{vx}Q_{vw})l_{vy} \right] \mathbf{e}_{43}$$

$$+ \left[-4(Q_{vy}Q_{my} + Q_{vz}Q_{mz})l_{vx} + 2(Q_{vy}Q_{mx} + Q_{vx}Q_{my} - Q_{vz}Q_{mw} - Q_{vw}Q_{mz})l_{vy} + 2(Q_{vz}Q_{mx} + Q_{vx}Q_{mz} + Q_{vy}Q_{mw} + Q_{vw}Q_{my})l_{vz} + (1 - 2Q_{vy}^2 - 2Q_{vz}^2)l_{mx} + 2(Q_{vx}Q_{vy} - Q_{vz}Q_{vw})l_{my} + 2(Q_{vz}Q_{vx} + Q_{vy}Q_{vw})l_{mz} \right] \mathbf{e}_{23}$$

$$+ \left[-4(Q_{vz}Q_{mz} + Q_{vx}Q_{mx})l_{vy} + 2(Q_{vz}Q_{my} + Q_{vy}Q_{mz} - Q_{vx}Q_{mw} - Q_{vw}Q_{mx})l_{vz} + 2(Q_{vx}Q_{my} + Q_{vy}Q_{mx} + Q_{vz}Q_{mw} + Q_{vw}Q_{mz})l_{vx} + (1 - 2Q_{vz}^2 - 2Q_{vx}^2)l_{my} + 2(Q_{vy}Q_{vz} - Q_{vx}Q_{vw})l_{mz} + 2(Q_{vx}Q_{vy} + Q_{vz}Q_{vw})l_{mx} \right] \mathbf{e}_{31}$$

$$+ \left[-4(Q_{vx}Q_{mx} + Q_{vy}Q_{my})l_{vz} + 2(Q_{vx}Q_{mz} + Q_{vz}Q_{mx} - Q_{vy}Q_{mw} - Q_{vw}Q_{my})l_{vx} + 2(Q_{vy}Q_{mz} + Q_{vz}Q_{my} + Q_{vx}Q_{mw} + Q_{vw}Q_{mx})l_{vy} + (1 - 2Q_{vx}^2 - 2Q_{vy}^2)l_{mz} + 2(Q_{vz}Q_{vx} - Q_{vy}Q_{vw})l_{mx} + 2(Q_{vy}Q_{vz} + 2Q_{vx}Q_{vw})l_{my} \right] \mathbf{e}_{12}$$

3x4 matrix transforming parametric line only needs 21 mads

GDC

Motor Transforming Plane

Can be reduced to 39 multiply-adds

$$\mathbf{Q} \vee \mathbf{g} \vee \underset{\sim}{\mathbf{Q}} = \left[(1 - 2Q_{vy}^2 - 2Q_{vz}^2)g_x + 2(Q_{vx}Q_{vy} - Q_{vz}Q_{vw})g_y + 2(Q_{vz}Q_{vx} + Q_{vy}Q_{vw})g_z \right] \mathbf{e}_{423}$$

$$+ \left[(1 - 2Q_{vz}^2 - 2Q_{vx}^2)g_y + 2(Q_{vy}Q_{vz} - Q_{vx}Q_{vw})g_z + 2(Q_{vx}Q_{vy} + Q_{vz}Q_{vw})g_x \right] \mathbf{e}_{431}$$

$$+ \left[(1 - 2Q_{vx}^2 - 2Q_{vy}^2)g_z + 2(Q_{vz}Q_{vx} - Q_{vy}Q_{vw})g_x + 2(Q_{vy}Q_{vz} + Q_{vx}Q_{vw})g_y \right] \mathbf{e}_{412}$$

$$+ \left[2(Q_{vy}Q_{mz} - Q_{vz}Q_{my} + Q_{vx}Q_{mw} - Q_{vw}Q_{mx})g_x + 2(Q_{vz}Q_{mx} - Q_{vx}Q_{mz} + Q_{vy}Q_{mw} - Q_{vw}Q_{my})g_y + 2(Q_{vx}Q_{my} - Q_{vy}Q_{mx} + Q_{vz}Q_{mw} - Q_{vw}Q_{mz})g_z + g_w \right] \mathbf{e}_{321}$$

3x4 matrix needs to be inverted (9 mads, 3 negs for ortho), then 12 mads to transform plane

GDC

Motor to Matrix

$$\mathbf{M} = \mathbf{A} + \mathbf{B} \qquad \mathbf{M}^{-1} = \mathbf{A} - \mathbf{B}$$

$$\mathbf{A} = \begin{bmatrix} 1 - 2(Q_{vy}^2 + Q_{vz}^2) & 2Q_{vx}Q_{vy} & 2Q_{vz}Q_{vx} & 2(Q_{vy}Q_{mz} - Q_{vz}Q_{my}) \\ 2Q_{vx}Q_{vy} & 1 - 2(Q_{vz}^2 + Q_{vx}^2) & 2Q_{vy}Q_{vz} & 2(Q_{vz}Q_{mx} - Q_{vx}Q_{mz}) \\ 2Q_{vz}Q_{vx} & 2Q_{vy}Q_{vz} & 1 - 2(Q_{vx}^2 + Q_{vy}^2) & 2(Q_{vx}Q_{my} - Q_{vy}Q_{mx}) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 & -2Q_{vz}Q_{vw} & 2Q_{vy}Q_{vw} & 2(Q_{vw}Q_{mx} - Q_{vx}Q_{mw}) \\ 2Q_{vz}Q_{vw} & 0 & -2Q_{vx}Q_{vw} & 2(Q_{vw}Q_{my} - Q_{vy}Q_{mw}) \\ -2Q_{vy}Q_{vw} & 2Q_{vx}Q_{vw} & 0 & 2(Q_{vw}Q_{mz} - Q_{vz}Q_{mw}) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Motor Composition

Multiply motors together, 48 multiply-adds

$$\mathbf{R} \veebar \mathbf{Q} = \left(Q_{vw}R_{vx} + Q_{vx}R_{vw} - Q_{vy}R_{vz} + Q_{vz}R_{vy} \right) \mathbf{e}_{41}$$
$$+ \left(Q_{vw}R_{vy} + Q_{vx}R_{vz} + Q_{vy}R_{vw} - Q_{vz}R_{vx} \right) \mathbf{e}_{42}$$
$$+ \left(Q_{vw}R_{vz} - Q_{vx}R_{vy} + Q_{vy}R_{vx} + Q_{vz}R_{vw} \right) \mathbf{e}_{43}$$
$$+ \left(Q_{vw}R_{vw} - Q_{vx}R_{vx} - Q_{vy}R_{vy} - Q_{vz}R_{vz} \right) \mathbb{1}$$
$$+ \left(Q_{mw}R_{vx} + Q_{mx}R_{vw} - Q_{my}R_{vz} + Q_{mz}R_{vy} + Q_{vw}R_{mx} + Q_{vx}R_{mw} - Q_{vy}R_{mz} + Q_{vz}R_{my} \right) \mathbf{e}_{23}$$
$$+ \left(Q_{mw}R_{vy} + Q_{mx}R_{vz} + Q_{my}R_{vw} - Q_{mz}R_{vx} + Q_{vw}R_{my} + Q_{vx}R_{mz} + Q_{vy}R_{mw} - Q_{vz}R_{mx} \right) \mathbf{e}_{31}$$
$$+ \left(Q_{mw}R_{vz} - Q_{mx}R_{vy} + Q_{my}R_{vx} + Q_{mz}R_{vw} + Q_{vw}R_{mz} - Q_{vx}R_{my} + Q_{vy}R_{mx} + Q_{vz}R_{mw} \right) \mathbf{e}_{12}$$
$$+ \left(Q_{mw}R_{vw} - Q_{mx}R_{vx} - Q_{my}R_{vy} - Q_{mz}R_{vz} + Q_{vw}R_{mw} - Q_{vx}R_{mx} - Q_{vy}R_{my} - Q_{vz}R_{mz} \right) \mathbf{1}$$

Composing two 3x4 matrices needs 39 multiply-adds

GDC

Matrix Advantages

Faster to transform objects

Faster to compose

Can read off origin and axis directions in transformed space

Motor Advantages

Smaller storage requirements
- Can get down to six floats

Inversion trivial
- It's just the reverse

Interpolates nicer

De-orthogonalizes slower

Reciprocal Transformations

Sandwich with geometric antiproduct preserves horizon

Sandwich with geometric product preserves origin

Two transformations always happening simultaneously

Switching products swaps operations happening in space and antispace

Reciprocal Rotation

When rotation happens in space, a reciprocal rotation happens in antispace

Dual of rotation axis preserved

Line through origin parallel to rotation axis preserved

Points follow orbits of constant eccentricity with respect to directrix given by dual of rotation axis

GDC
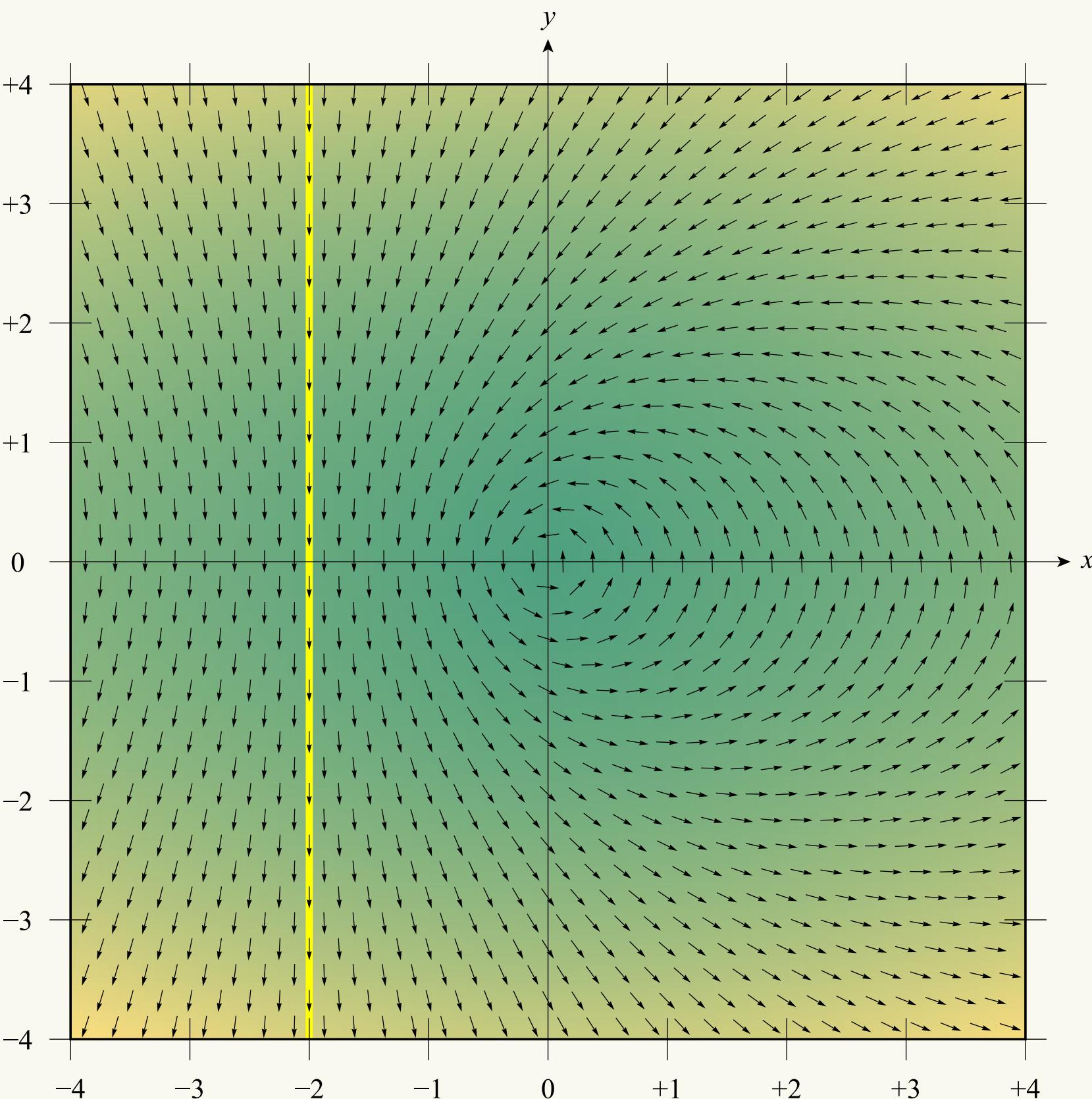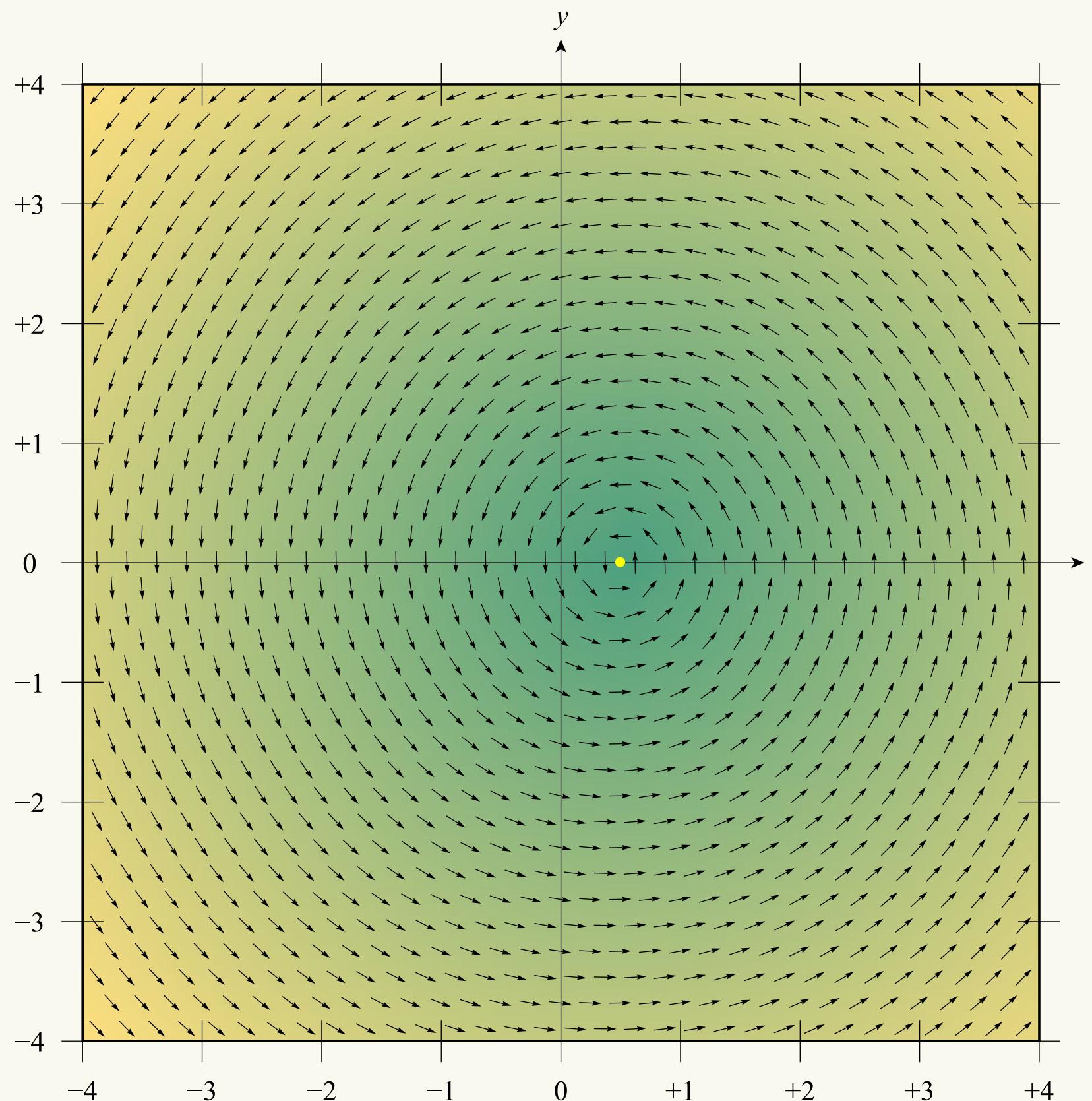
Reciprocal Translation

When translation happens in space, a reciprocal translation happens in antispace

Performs perspective projection!

Translation distance corresponds to focal length

Unfortunately, doesn't project in the way needed by GPUs
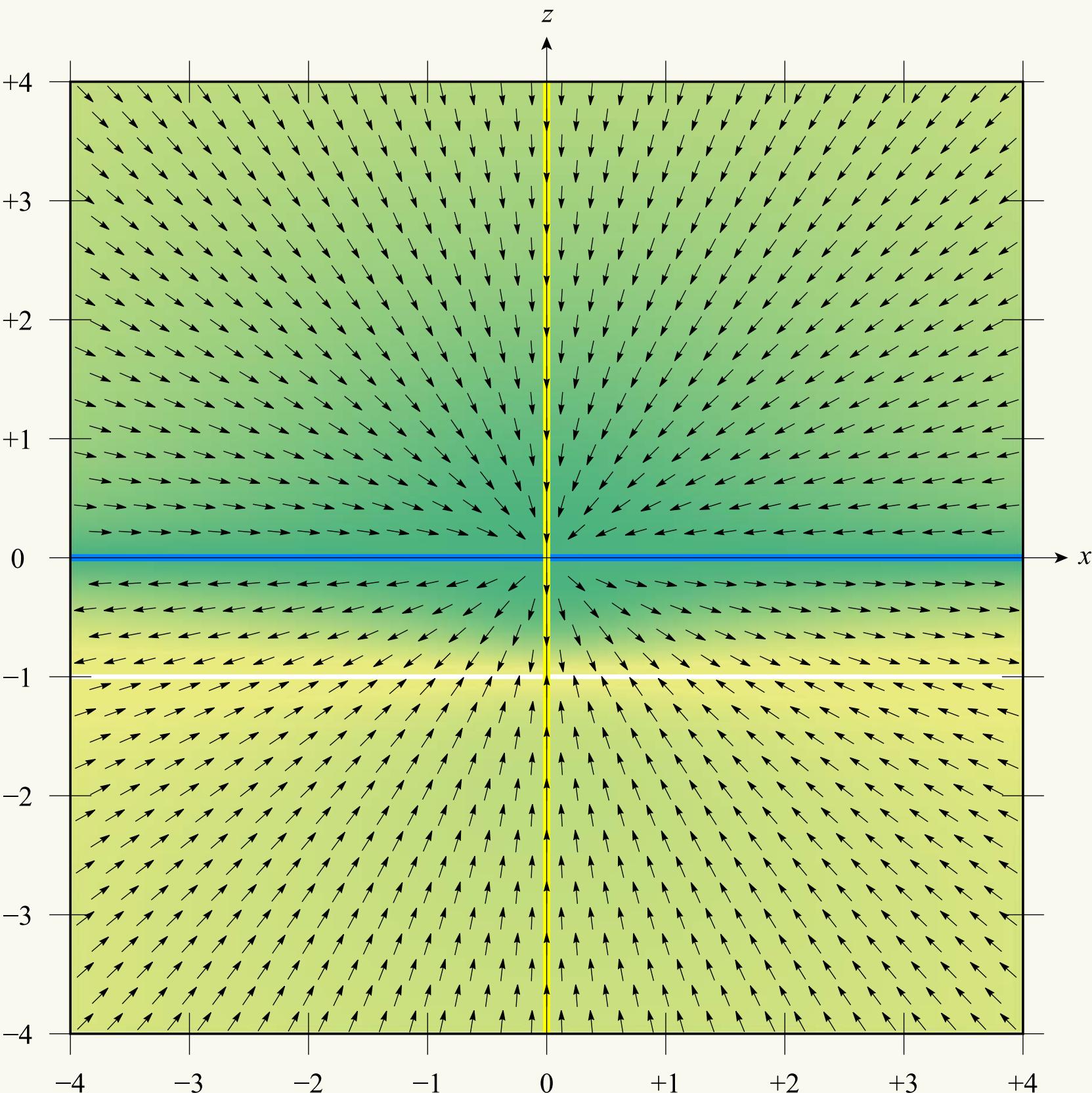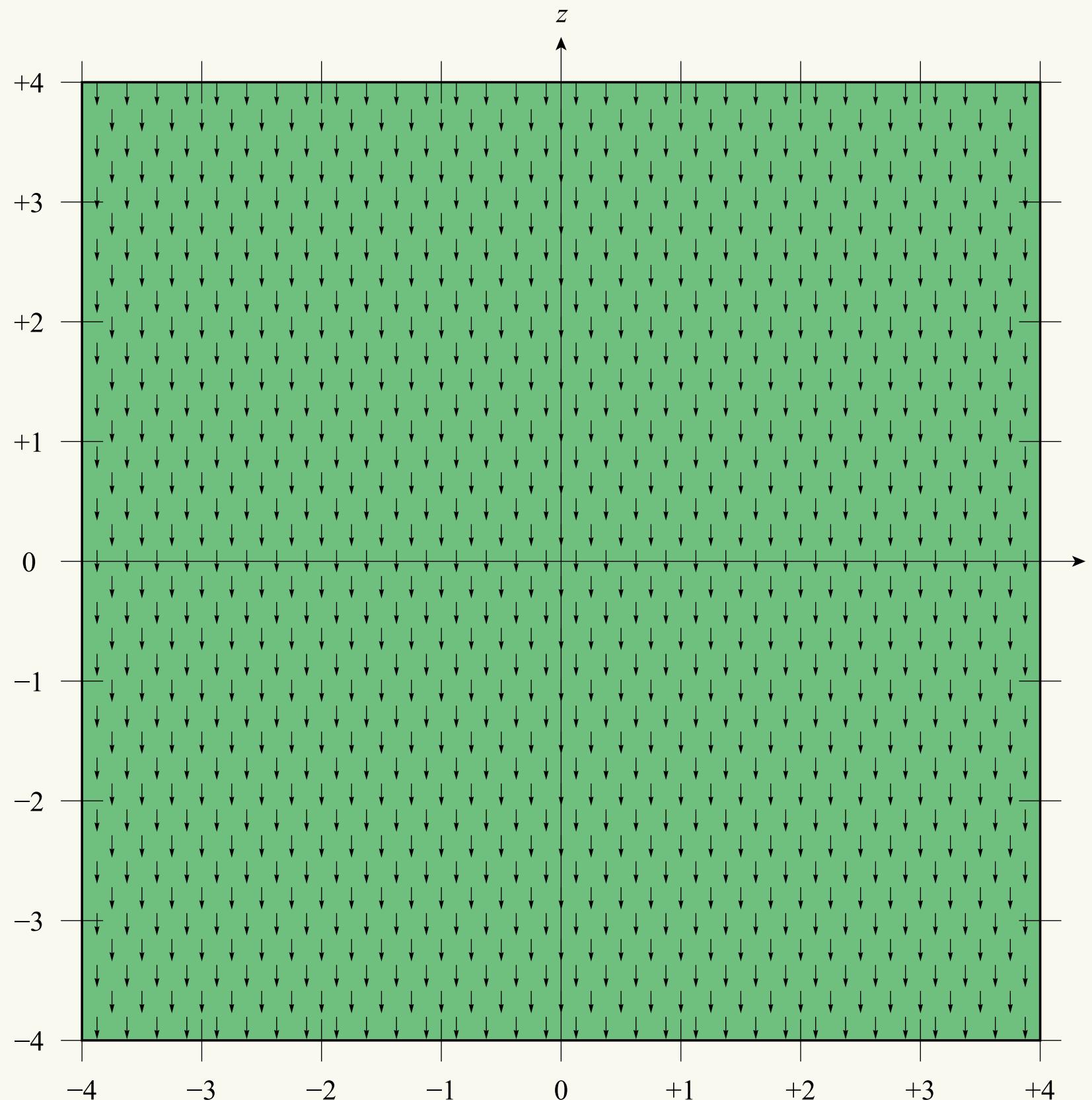
Transformation Groups
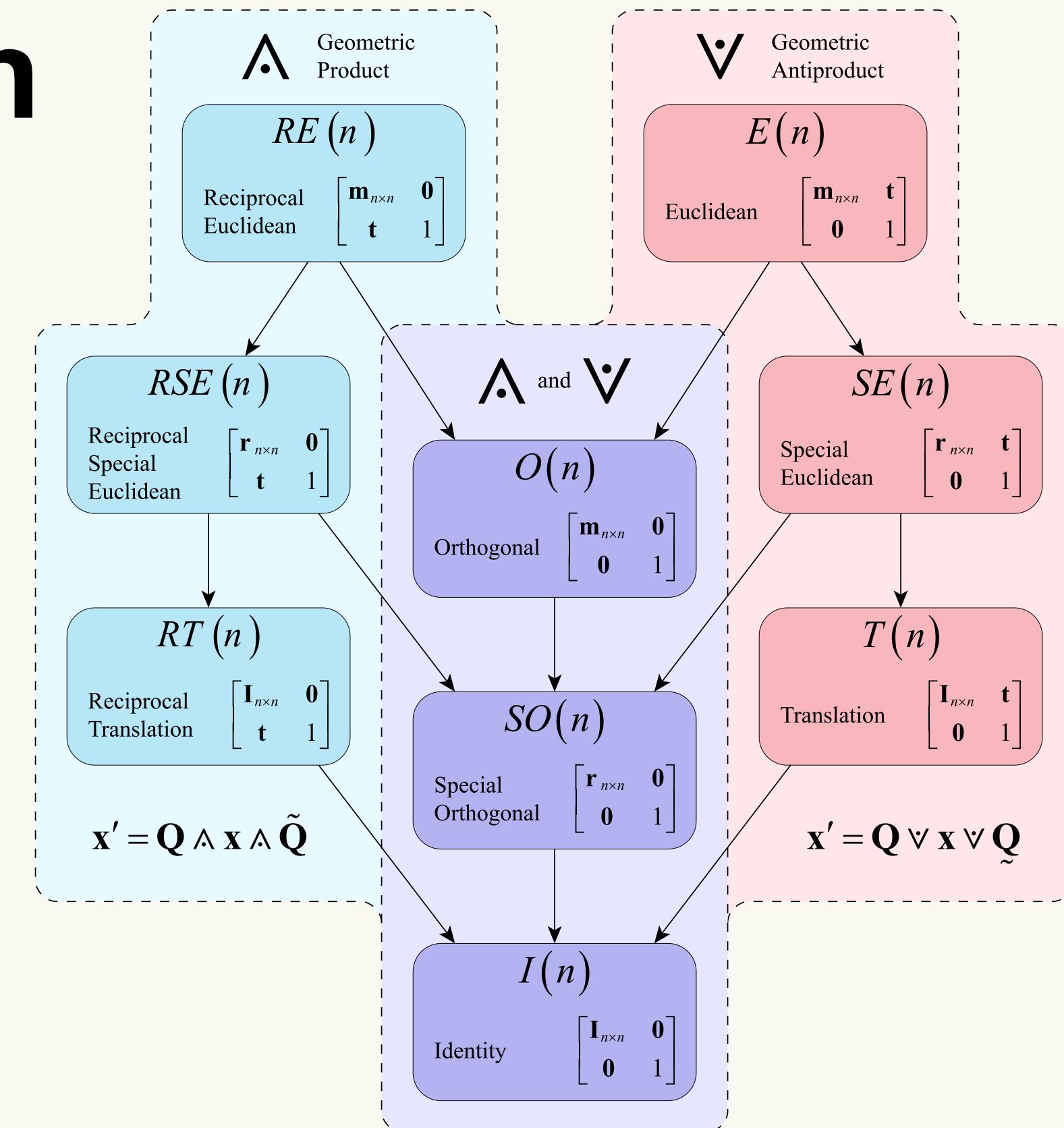
Conformal Geometric Algebra

Projective algebra with two extra dimensions

Contains all of rigid geometric algebra

- Flat points, lines, planes
- Rotations, translations, screw transformations

Has same projection as RGA, plus another stereographic projection

Conformal Geometric Algebra

Contains round points, dipoles, circles, spheres

Can perform dilations, inversions in spheres, and all the conformal transformations derived from those

The Horosphere
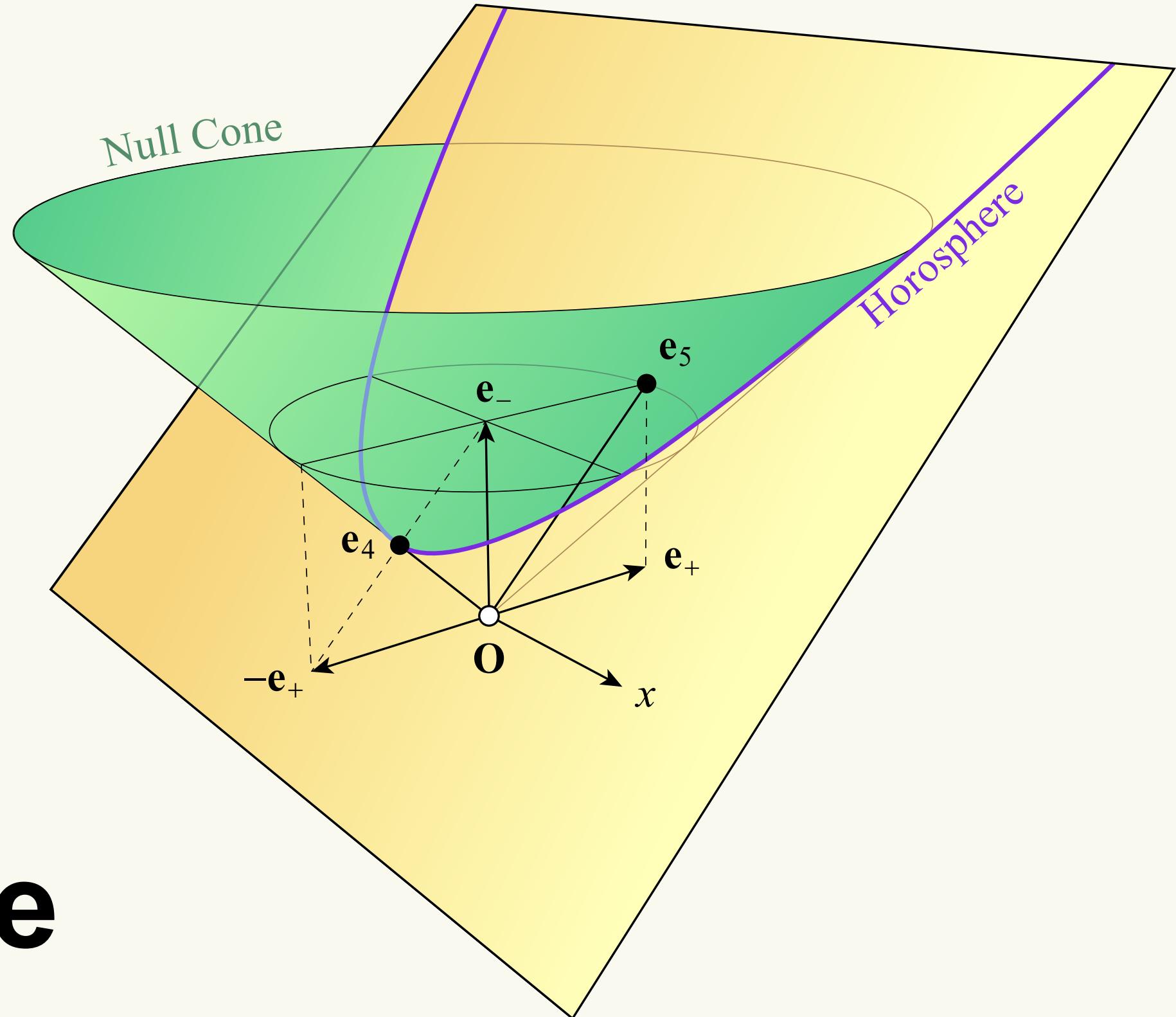
Origin and Infinity

Five basis elements

Origin = $\mathbf{e}_4$

Infinity = $\mathbf{e}_5$

Object Types in CGA

Flat Object	Representation
Flat point **p**	
Line *l*	
Plane **g**	

Round Object	Real	Imaginary
Round point **a**		
Dipole **d**		
Circle **c**		
Sphere **s**		

Joining Points
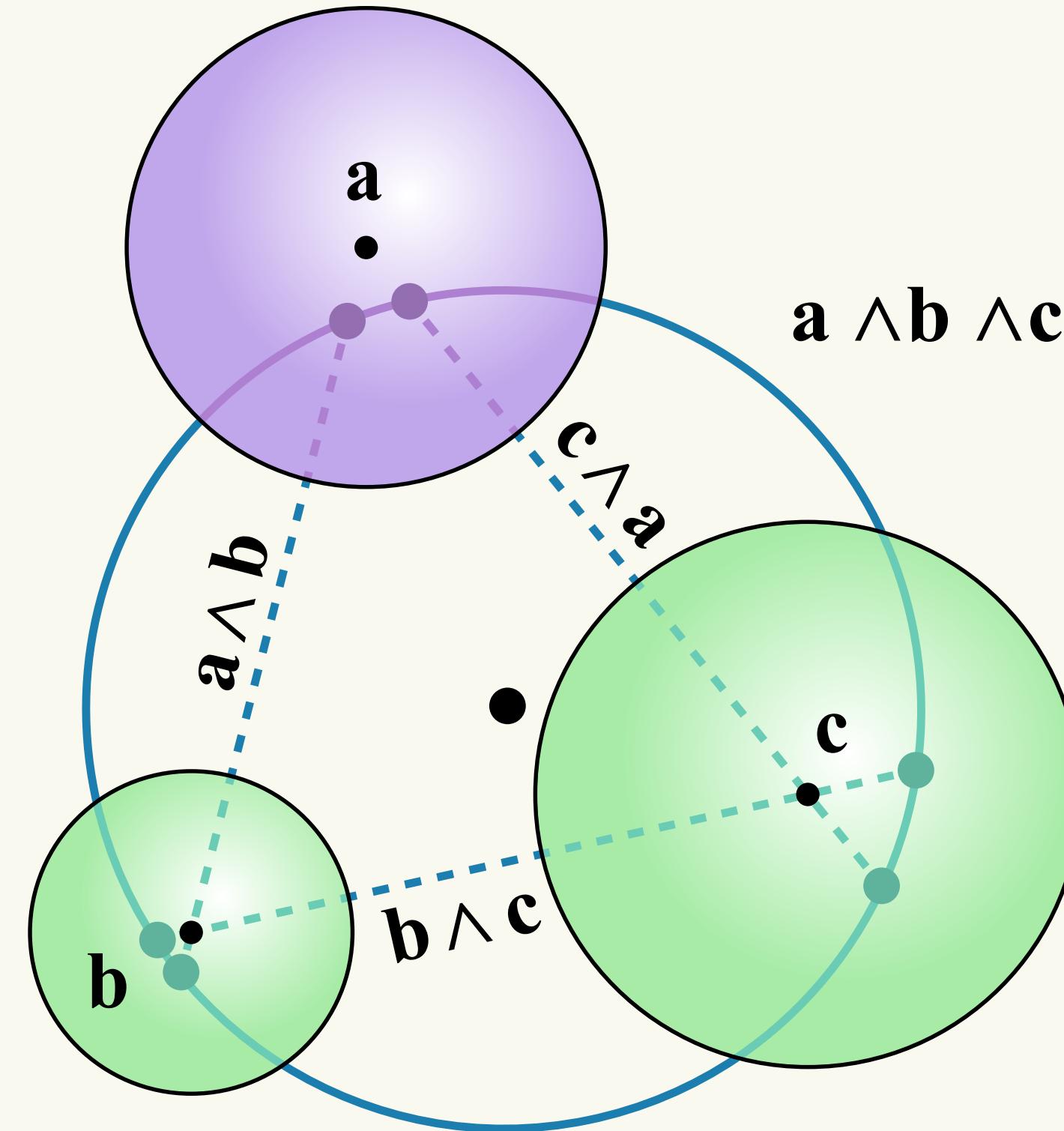
Flats

Flat Point **p**	
(Bivector) **0D**	
$\mathbf{p} = p_x\mathbf{e}_{15} + p_y\mathbf{e}_{25} + p_z\mathbf{e}_{35} + p_w\mathbf{e}_{45}$	
Dual	$\mathbf{p}^* = -p_x\mathbf{e}_{235} - p_y\mathbf{e}_{315} - p_z\mathbf{e}_{125} + p_w\mathbf{e}_{321}$
Attitude	$\mathrm{att}(\mathbf{p}) = \mathbf{p} \vee \mathbf{e}_{3215} = p_w\mathbf{e}_5$
Flat Bulk	$\mathbf{p}_\blacksquare = p_x\mathbf{e}_{15} + p_y\mathbf{e}_{25} + p_z\mathbf{e}_{35}$
Flat Weight	$\mathbf{p}_\square = p_w\mathbf{e}_{45}$
Position Norm	$\dfrac{\|\mathbf{p}\|_\blacksquare}{\|\mathbf{p}\|_\square} = \sqrt{\dfrac{p_x^2 + p_y^2 + p_z^2}{p_w^2}}$

Flat Line **l**	
(Trivector) **1D**	
$\boldsymbol{l} = l_{vx}\mathbf{e}_{415} + l_{vy}\mathbf{e}_{425} + l_{vz}\mathbf{e}_{435}$ $+ l_{mx}\mathbf{e}_{235} + l_{my}\mathbf{e}_{315} + l_{mz}\mathbf{e}_{125}$	$l_{vx}l_{mx} + l_{vy}l_{my} + l_{vz}l_{mz} = 0$
Dual	$\boldsymbol{l}^* = l_{vx}\mathbf{e}_{23} + l_{vy}\mathbf{e}_{31} + l_{vz}\mathbf{e}_{12} + l_{mx}\mathbf{e}_{15} + l_{my}\mathbf{e}_{25} + l_{mz}\mathbf{e}_{35}$
Attitude	$\mathrm{att}(\boldsymbol{l}) = \boldsymbol{l} \vee \mathbf{e}_{3215} = l_{vx}\mathbf{e}_{15} + l_{vy}\mathbf{e}_{25} + l_{vz}\mathbf{e}_{35}$
Flat Bulk	$\boldsymbol{l}_\blacksquare = l_{mx}\mathbf{e}_{235} + l_{my}\mathbf{e}_{315} + l_{mz}\mathbf{e}_{125}$ (moment)
Flat Weight	$\boldsymbol{l}_\square = l_{vx}\mathbf{e}_{415} + l_{vy}\mathbf{e}_{425} + l_{vz}\mathbf{e}_{435}$ (direction)
Position Norm	$\dfrac{\|\boldsymbol{l}\|_\blacksquare}{\|\boldsymbol{l}\|_\square} = \sqrt{\dfrac{l_{mx}^2 + l_{my}^2 + l_{mz}^2}{l_{vx}^2 + l_{vy}^2 + l_{vz}^2}}$

Flat Plane **g**	
(Quadrivector) **2D**	
$\mathbf{g} = g_x\mathbf{e}_{4235} + g_y\mathbf{e}_{4315} + g_z\mathbf{e}_{4125} + g_w\mathbf{e}_{3215}$	
Dual	$\mathbf{g}^* = -g_x\mathbf{e}_1 - g_y\mathbf{e}_2 - g_z\mathbf{e}_3 + g_w\mathbf{e}_5$
Attitude	$\mathrm{att}(\mathbf{g}) = \mathbf{g} \vee \mathbf{e}_{3215} = g_x\mathbf{e}_{235} + g_y\mathbf{e}_{315} + g_z\mathbf{e}_{125}$
Flat Bulk	$\mathbf{g}_\blacksquare = g_w\mathbf{e}_{3215}$ (position)
Flat Weight	$\mathbf{g}_\square = g_x\mathbf{e}_{4235} + g_y\mathbf{e}_{4315} + g_z\mathbf{e}_{4125}$ (normal)
Position Norm	$\dfrac{\|\mathbf{g}\|_\blacksquare}{\|\mathbf{g}\|_\square} = \sqrt{\dfrac{g_w^2}{g_x^2 + g_y^2 + g_z^2}}$
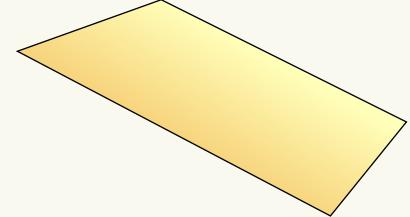
Round Point

Round Point **a** (Vector) **0D**	$r^2 > 0$ $r^2 < 0$	$\mathbf{a} = p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3 + \mathbf{e}_4$ $+ \dfrac{p^2 + r^2}{2} \mathbf{e}_5$ $\mathbf{p} = \text{position}$ $r = \text{radius}$

$\mathbf{a} = a_x \mathbf{e}_1 + a_y \mathbf{e}_2 + a_z \mathbf{e}_3 + a_w \mathbf{e}_4 + a_u \mathbf{e}_5$ Carrier Point Infinity $(\text{when } a_x = a_y = a_z = a_w = 0)$	Attitude $\text{att}(\mathbf{a}) = \mathbf{a} \vee \mathbf{e}_{3215}$ $= a_w \mathbf{1}$

Center	$\text{cen}(\mathbf{a}) = -\text{acr}(\mathbf{a}) \vee \mathbf{a}$ $= a_x a_w \mathbf{e}_1 + a_y a_w \mathbf{e}_2 + a_z a_w \mathbf{e}_3 + a_w^2 \mathbf{e}_4 + a_w a_u \mathbf{e}_5$	(identity)
Container	$\text{con}(\mathbf{a}) = \text{car}(\mathbf{a})^* \wedge \mathbf{a}$ $= -a_w^2 \mathbf{e}_{1234} + a_x a_w \mathbf{e}_{4235} + a_y a_w \mathbf{e}_{4315} + a_z a_w \mathbf{e}_{4125} + \left(a_w a_u - a_x^2 - a_y^2 - a_z^2 \right) \mathbf{e}_{3215}$	
Dual	$\mathbf{a}^* = -a_w \mathbf{e}_{1234} + a_x \mathbf{e}_{4235} + a_y \mathbf{e}_{4315} + a_z \mathbf{e}_{4125} - a_u \mathbf{e}_{3215}$	
Carrier	$\text{car}(\mathbf{a}) = \mathbf{a} \wedge \mathbf{e}_5 = a_x \mathbf{e}_{15} + a_y \mathbf{e}_{25} + a_z \mathbf{e}_{35} + a_w \mathbf{e}_{45}$	(flat point)
Anticarrier	$\text{acr}(\mathbf{a}) = \mathbf{a}^* \wedge \mathbf{e}_5 = -a_w \mathbb{1}$	(volume)
Round Bulk	$\mathbf{a}_\bullet = a_x \mathbf{e}_1 + a_y \mathbf{e}_2 + a_z \mathbf{e}_3$	Center Norm	$\dfrac{\lVert \mathbf{a} \rVert_\circ}{\lVert \mathbf{a} \rVert_\circ} = \sqrt{\dfrac{a_x^2 + a_y^2 + a_z^2}{a_w^2}}$
Round Weight	$\mathbf{a}_\circ = a_w \mathbf{e}_4$		
Flat Bulk	$\mathbf{a}_\blacksquare = a_u \mathbf{e}_5$	Radius Norm	$\dfrac{\lVert \mathbf{a} \rVert_\circ}{\lVert \mathbf{a} \rVert_\circ} = \sqrt{\dfrac{2 a_w a_u - a_x^2 - a_y^2 - a_z^2}{a_w^2}}$
Flat Weight	$\mathbf{a}_\square = 0$		

Dipole

| Dipole $\mathbf{d}$ (Bivector) **1D** | | $\mathbf{d} = n_x\mathbf{e}_{41} + n_y\mathbf{e}_{42} + n_z\mathbf{e}_{43} + (p_yn_z - p_zn_y)\mathbf{e}_{23} + (p_zn_x - p_xn_z)\mathbf{e}_{31} + (p_xn_y - p_yn_x)\mathbf{e}_{12}$ $+ (\mathbf{p}\cdot\mathbf{n})(p_x\mathbf{e}_{15} + p_y\mathbf{e}_{25} + p_z\mathbf{e}_{35} + \mathbf{e}_{45}) - \dfrac{p^2+r^2}{2}(n_x\mathbf{e}_{15} + n_y\mathbf{e}_{25} + n_z\mathbf{e}_{35})$ | $\mathbf{p} = (p_x, p_y, p_z) = \text{center}$ $\mathbf{n} = (n_x, n_y, n_z) = \text{direction}$ $r = \text{radius}$ |

Anticarrier Normal · Anticarrier Position

$\mathbf{d} = d_{vx}\mathbf{e}_{41} + d_{vy}\mathbf{e}_{42} + d_{vz}\mathbf{e}_{43} + d_{mx}\mathbf{e}_{23} + d_{my}\mathbf{e}_{31} + d_{mz}\mathbf{e}_{12} + d_{px}\mathbf{e}_{15} + d_{py}\mathbf{e}_{25} + d_{pz}\mathbf{e}_{35} + d_{pw}\mathbf{e}_{45}$
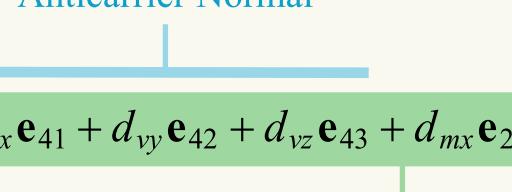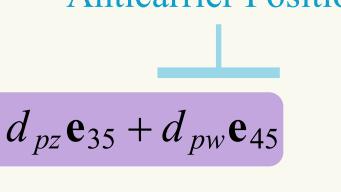
Carrier Line — Flat Point (when $d_{vz} = d_{vy} = d_{vz} = d_{mx} = d_{my} = d_{mz} = 0$)

Anticarrier · Container · Carrier · r · $\mathbf{p}$ · $\mathbf{n}$
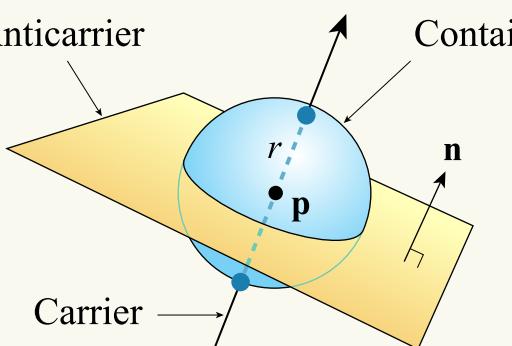
$d_{py}d_{vz} - d_{pz}d_{vy} - d_{pw}d_{mx} = 0$
$d_{pz}d_{vx} - d_{px}d_{vz} - d_{pw}d_{my} = 0$
$d_{px}d_{vy} - d_{py}d_{vx} - d_{pw}d_{mz} = 0$
$d_{px}d_{mx} + d_{py}d_{my} + d_{pz}d_{mz} = 0$
$d_{vx}d_{mx} + d_{vy}d_{my} + d_{vz}d_{mz} = 0$

Center	$\text{cen}(\mathbf{d}) = -\text{acr}(\mathbf{d})\vee\mathbf{d} = (d_{vy}d_{mz} - d_{vz}d_{my} + d_{vx}d_{pw})\mathbf{e}_1 + (d_{vz}d_{mx} - d_{vx}d_{mz} + d_{vy}d_{pw})\mathbf{e}_2 + (d_{vx}d_{my} - d_{vy}d_{mx} + d_{vz}d_{pw})\mathbf{e}_3 + (d_{vx}^2 + d_{vy}^2 + d_{vz}^2)\mathbf{e}_4 + (d_{pw}^2 - d_{vx}d_{px} - d_{vy}d_{py} - d_{vz}d_{pz})\mathbf{e}_5$
Dual	$\mathbf{d}^* = -d_{vx}\mathbf{e}_{423} - d_{vy}\mathbf{e}_{431} - d_{vz}\mathbf{e}_{412} + d_{pw}\mathbf{e}_{321} - d_{mx}\mathbf{e}_{415} - d_{my}\mathbf{e}_{425} - d_{mz}\mathbf{e}_{435} - d_{px}\mathbf{e}_{235} - d_{py}\mathbf{e}_{315} - d_{pz}\mathbf{e}_{125}$
Carrier	$\text{car}(\mathbf{d}) = \mathbf{d}\wedge\mathbf{e}_5 = d_{vx}\mathbf{e}_{415} + d_{vy}\mathbf{e}_{425} + d_{vz}\mathbf{e}_{435} + d_{mx}\mathbf{e}_{235} + d_{my}\mathbf{e}_{315} + d_{mz}\mathbf{e}_{125}$ (flat line)
Anticarrier	$\text{acr}(\mathbf{d}) = \mathbf{d}^*\wedge\mathbf{e}_5 = -d_{vx}\mathbf{e}_{4235} - d_{vy}\mathbf{e}_{4315} - d_{vz}\mathbf{e}_{4125} + d_{pw}\mathbf{e}_{3215}$ (flat plane)

Container:
$\text{con}(\mathbf{d}) = \text{car}(\mathbf{d})^*\wedge\mathbf{d} = -(d_{vx}^2 + d_{vy}^2 + d_{vz}^2)\mathbf{e}_{1234}$
$+ (d_{vy}d_{mz} - d_{vz}d_{my} + d_{vx}d_{pw})\mathbf{e}_{4235}$
$+ (d_{vz}d_{mx} - d_{vx}d_{mz} + d_{vy}d_{pw})\mathbf{e}_{4315}$
$+ (d_{vx}d_{my} - d_{vy}d_{mx} + d_{vz}d_{pw})\mathbf{e}_{4125}$
$- (d_{mx}^2 + d_{my}^2 + d_{mz}^2 + d_{vx}d_{px} + d_{vy}d_{py} + d_{vz}d_{pz})\mathbf{e}_{3215}$

Round Bulk	$\mathbf{d}_\bullet = d_{mx}\mathbf{e}_{23} + d_{my}\mathbf{e}_{31} + d_{mz}\mathbf{e}_{12}$	Attitude	$\text{att}(\mathbf{d}) = \mathbf{d}\vee\mathbf{e}_{3215}$ $= d_{vx}\mathbf{e}_1 + d_{vy}\mathbf{e}_2 + d_{vz}\mathbf{e}_3 + d_{pw}\mathbf{e}_5$
Round Weight	$\mathbf{d}_\circ = d_{vx}\mathbf{e}_{41} + d_{vy}\mathbf{e}_{42} + d_{vz}\mathbf{e}_{43}$		
Flat Bulk	$\mathbf{d}_\blacksquare = d_{px}\mathbf{e}_{15} + d_{py}\mathbf{e}_{25} + d_{pz}\mathbf{e}_{35}$	Center Norm	$\dfrac{\|\mathbf{d}\|_\odot}{\|\mathbf{d}\|_\circ} = \sqrt{\dfrac{d_{mx}^2 + d_{my}^2 + d_{mz}^2 + d_{pw}^2}{d_{vx}^2 + d_{vy}^2 + d_{vz}^2}}$
Flat Weight	$\mathbf{d}_\square = d_{pw}\mathbf{e}_{45}$		

Radius Norm: $\dfrac{\|\mathbf{d}\|_\odot}{\|\mathbf{d}\|_\circ} = \sqrt{\dfrac{d_{pw}^2 - d_{mx}^2 - d_{my}^2 - d_{mz}^2 - 2(d_{px}d_{vx} + d_{py}d_{vy} + d_{pz}d_{vz})}{d_{vx}^2 + d_{vy}^2 + d_{vz}^2}}$

GDC

Circle

Circle **c** (Trivector) **2D**		$\mathbf{c} = n_x \mathbf{e}_{423} + n_y \mathbf{e}_{431} + n_z \mathbf{e}_{412} + \left(p_y n_z - p_z n_y \right) \mathbf{e}_{415} + \left(p_z n_x - p_x n_z \right) \mathbf{e}_{425} + \left(p_x n_y - p_y n_x \right) \mathbf{e}_{435}$ $+ \left(\mathbf{p} \cdot \mathbf{n} \right) \left(p_x \mathbf{e}_{235} + p_y \mathbf{e}_{315} + p_z \mathbf{e}_{125} - \mathbf{e}_{321} \right) - \dfrac{p^2 - r^2}{2} \left(n_x \mathbf{e}_{235} + n_y \mathbf{e}_{315} + n_z \mathbf{e}_{125} \right)$	$\mathbf{p} = \left(p_x, p_y, p_z \right) = \text{center}$ $\mathbf{n} = \left(n_x, n_y, n_z \right) = \text{normal}$ $r = \text{radius}$

Anticarrier Direction   **Anticarrier Moment**

$\mathbf{c} = c_{gx} \mathbf{e}_{423} + c_{gy} \mathbf{e}_{431} + c_{gz} \mathbf{e}_{412} + c_{gw} \mathbf{e}_{321} + c_{vx} \mathbf{e}_{415} + c_{vy} \mathbf{e}_{425} + c_{vz} \mathbf{e}_{435} + c_{mx} \mathbf{e}_{235} + c_{my} \mathbf{e}_{315} + c_{mz} \mathbf{e}_{125}$

Carrier Plane

Flat Line
(when $c_{gx} = c_{gy} = c_{gz} = c_{gw} = 0$)

Carrier   Container   Anticarrier
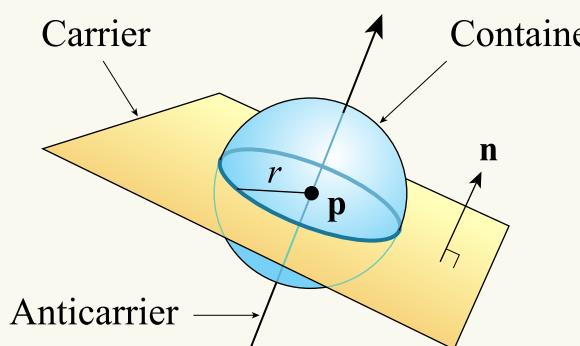
$c_{gy} c_{mz} - c_{gz} c_{my} - c_{gw} c_{vx} = 0$
$c_{gz} c_{mx} - c_{gx} c_{mz} - c_{gw} c_{vy} = 0$
$c_{gx} c_{my} - c_{gy} c_{mx} - c_{gw} c_{vz} = 0$
$c_{vx} c_{mx} + c_{vy} c_{my} + c_{vz} c_{mz} = 0$
$c_{gx} c_{vx} + c_{gy} c_{vy} + c_{gz} c_{vz} = 0$

Center	$\mathrm{cen}\left(\mathbf{c} \right) = -\,\mathrm{acr}\left(\mathbf{c} \right) \vee \mathbf{c} = \left(c_{gy} c_{vz} - c_{gz} c_{vy} - c_{gx} c_{gw} \right) \mathbf{e}_1 + \left(c_{gz} c_{vx} - c_{gx} c_{vz} - c_{gy} c_{gw} \right) \mathbf{e}_2 + \left(c_{gx} c_{vy} - c_{gy} c_{vx} - c_{gz} c_{gw} \right) \mathbf{e}_3 + \left(c_{gx}^2 + c_{gy}^2 + c_{gz}^2 \right) \mathbf{e}_4 + \left(c_{vx}^2 + c_{vy}^2 + c_{vz}^2 + c_{gx} c_{mx} + c_{gy} c_{my} + c_{gz} c_{mz} \right) \mathbf{e}_5$			
Dual	$\mathbf{c}^{*} = c_{gx} \mathbf{e}_{41} + c_{gy} \mathbf{e}_{42} + c_{gz} \mathbf{e}_{43} + c_{vx} \mathbf{e}_{23} + c_{vy} \mathbf{e}_{31} + c_{vz} \mathbf{e}_{12} + c_{mx} \mathbf{e}_{15} + c_{my} \mathbf{e}_{25} + c_{mz} \mathbf{e}_{35} - c_{gw} \mathbf{e}_{45}$			$\mathrm{con}\left(\mathbf{c} \right) = \mathrm{car}\left(\mathbf{c} \right)^{*} \wedge \mathbf{c} = -\left(c_{gx}^2 + c_{gy}^2 + c_{gz}^2 \right) \mathbf{e}_{1234}$
Carrier	$\mathrm{car}\left(\mathbf{c} \right) = \mathbf{c} \wedge \mathbf{e}_5 = c_{gx} \mathbf{e}_{4235} + c_{gy} \mathbf{e}_{4315} + c_{gz} \mathbf{e}_{4125} + c_{gw} \mathbf{e}_{3215}$	(flat plane)		$+ \left(c_{gy} c_{vz} - c_{gz} c_{vy} - c_{gx} c_{gw} \right) \mathbf{e}_{4235}$
Anticarrier	$\mathrm{acr}\left(\mathbf{c} \right) = \mathbf{c}^{*} \wedge \mathbf{e}_5 = c_{gx} \mathbf{e}_{415} + c_{gy} \mathbf{e}_{425} + c_{gz} \mathbf{e}_{435} + c_{vx} \mathbf{e}_{235} + c_{vy} \mathbf{e}_{315} + c_{vz} \mathbf{e}_{125}$	(flat line)		$+ \left(c_{gz} c_{vx} - c_{gx} c_{vz} - c_{gy} c_{gw} \right) \mathbf{e}_{4315}$
Round Bulk	$\mathbf{c}_{\bullet} = c_{gw} \mathbf{e}_{321}$	Attitude	$\mathrm{att}\left(\mathbf{c} \right) = \mathbf{c} \vee \mathbf{e}_{3215} = c_{gx} \mathbf{e}_{23} + c_{gy} \mathbf{e}_{31} + c_{gz} \mathbf{e}_{12}$ $+ c_{vx} \mathbf{e}_{15} + c_{vy} \mathbf{e}_{25} + c_{vz} \mathbf{e}_{35}$		$+ \left(c_{gx} c_{vy} - c_{gy} c_{vx} - c_{gz} c_{gw} \right) \mathbf{e}_{4125}$
Round Weight	$\mathbf{c}_{\bigcirc} = c_{gx} \mathbf{e}_{423} + c_{gy} \mathbf{e}_{431} + c_{gz} \mathbf{e}_{412}$			$+ \left(c_{gx} c_{mx} + c_{gy} c_{my} + c_{gz} c_{mz} - c_{gw}^2 \right) \mathbf{e}_{3215}$
Flat Bulk	$\mathbf{c}_{\blacksquare} = c_{mx} \mathbf{e}_{235} + c_{my} \mathbf{e}_{315} + c_{mz} \mathbf{e}_{125}$	Center Norm	$\dfrac{\|\mathbf{c}\|_{\odot}}{\|\mathbf{c}\|_{\bigcirc}} = \sqrt{\dfrac{c_{gw}^2 + c_{vx}^2 + c_{vy}^2 + c_{vz}^2}{c_{gx}^2 + c_{gy}^2 + c_{gz}^2}}$	Radius Norm	$\dfrac{\|\mathbf{c}\|_{\odot}}{\|\mathbf{c}\|_{\bigcirc}} = \sqrt{\dfrac{c_{vx}^2 + c_{vy}^2 + c_{vz}^2 - c_{gw}^2 + 2\left(c_{gx} c_{mx} + c_{gy} c_{my} + c_{gz} c_{mz} \right)}{c_{gx}^2 + c_{gy}^2 + c_{gz}^2}}$
Flat Weight	$\mathbf{c}_{\square} = c_{vx} \mathbf{e}_{415} + c_{vy} \mathbf{e}_{425} + c_{vz} \mathbf{e}_{435}$			

Sphere

Sphere **s** (Quadrivector) 3D		$$\mathbf{s} = p_x \mathbf{e}_{4235} + p_y \mathbf{e}_{4315} + p_z \mathbf{e}_{4125} - \mathbf{e}_{1234}$$ $$-\frac{p^2 - r^2}{2}\mathbf{e}_{3215}$$ $\mathbf{p}$ = center r = radius
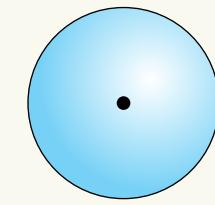
$$\mathbf{s} = \underbrace{s_u \mathbf{e}_{1234}}_{\text{Carrier Space}} + \underbrace{s_x \mathbf{e}_{4235} + s_y \mathbf{e}_{4315} + s_z \mathbf{e}_{4125} + s_w \mathbf{e}_{3215}}_{\substack{\text{Flat Plane} \\ (\text{when } s_u = 0)}}$$

	Attitude
	$\text{att}(\mathbf{s}) = \mathbf{s} \vee \mathbf{e}_{3215}$ $= s_u \mathbf{e}_{321} + s_x \mathbf{e}_{235}$ $+ s_y \mathbf{e}_{315} + s_z \mathbf{e}_{125}$

Center	$\text{cen}(\mathbf{s}) = -\text{acr}(\mathbf{s}) \vee \mathbf{s}$ $= -s_x s_u \mathbf{e}_1 - s_y s_u \mathbf{e}_2 - s_z s_u \mathbf{e}_3 + s_u^2 \mathbf{e}_4 + \left(s_x^2 + s_y^2 + s_z^2 - s_w s_u\right)\mathbf{e}_5$	
Container	$\text{con}(\mathbf{s}) = \text{car}(\mathbf{s})^* \wedge \mathbf{s}$ $= -s_u^2 \mathbf{e}_{1234} - s_x s_u \mathbf{e}_{4235} - s_y s_u \mathbf{e}_{4315} - s_z s_u \mathbf{e}_{4125} - s_w s_u \mathbf{e}_{3215}$	(identity)
Dual	$\mathbf{s}^* = -s_x \mathbf{e}_1 - s_y \mathbf{e}_2 - s_z \mathbf{e}_3 + s_u \mathbf{e}_4 + s_w \mathbf{e}_5$	
Carrier	$\text{car}(\mathbf{s}) = \mathbf{s} \wedge \mathbf{e}_5 = s_u \mathbb{1}$	(volume)
Anticarrier	$\text{acr}(\mathbf{s}) = \mathbf{s}^* \wedge \mathbf{e}_5 = -s_x \mathbf{e}_{15} - s_y \mathbf{e}_{25} - s_z \mathbf{e}_{35} + s_u \mathbf{e}_{45}$	(flat point)

Round Bulk	$\mathbf{s}_\bullet = 0$	Center Norm	$\dfrac{\|\mathbf{s}\|_\odot}{\|\mathbf{s}\|_\bigcirc} = \sqrt{\dfrac{s_x^2 + s_y^2 + s_z^2}{s_u^2}}$
Round Weight	$\mathbf{s}_\bigcirc = s_u \mathbf{e}_{1234}$		
Flat Bulk	$\mathbf{s}_\blacksquare = s_w \mathbf{e}_{3215}$	Radius Norm	$\dfrac{\|\mathbf{s}\|_\odot}{\|\mathbf{s}\|_\bigcirc} = \sqrt{\dfrac{s_x^2 + s_y^2 + s_z^2 - 2s_w s_u}{s_u^2}}$
Flat Weight	$\mathbf{s}_\square = s_x \mathbf{e}_{4235} + s_y \mathbf{e}_{4315} + s_z \mathbf{e}_{4125}$		

Round and Flat Parts

Round part = all components without $\mathbf{e}_5$ factor

Flat part = all components with $\mathbf{e}_5$ factor

If round part zero, then it's a flat object

Norms

Two major norms giving radius and center magnitude

Radius norm: $\|\mathbf{x}\|_{\oslash} = \sqrt{\mathbf{x} \circ \underset{\sim}{\mathbf{x}}} = \sqrt{-\mathbf{x} \bullet \tilde{\mathbf{x}}}$

Center norm: $\|\mathbf{x}\|_{\odot} = \sqrt{\mathbf{x} \bullet \overset{\leftrightarrow}{\tilde{\mathbf{x}}}} = \sqrt{-\mathbf{x} \circ \underset{\leftrightarrow}{\mathbf{x}}}$

Reflected Reverse

Reflected reverse = $\overleftrightarrow{\tilde{\mathbf{x}}}$

Same as reverse except $\mathbf{e}_5$ factor is negated

Used only in center norm

Norms

Type	Center Norm	Radius Norm
Round point $\mathbf{a}$	$\|\mathbf{a}\|_{\odot} = \sqrt{a_x^2 + a_y^2 + a_z^2}$	$\|\mathbf{a}\|_{\ominus} = \sqrt{2a_w a_u - a_x^2 - a_y^2 - a_z^2}$
Dipole $\mathbf{d}$	$\|\mathbf{d}\|_{\odot} = \sqrt{d_{mx}^2 + d_{my}^2 + d_{mz}^2 + d_{pw}^2}$	$\|\mathbf{d}\|_{\ominus} = \sqrt{d_{pw}^2 - d_{mx}^2 - d_{my}^2 - d_{mz}^2 - 2\left(d_{px}d_{vx} + d_{py}d_{vy} + d_{pz}d_{vz}\right)}$
Circle $\mathbf{c}$	$\|\mathbf{c}\|_{\odot} = \sqrt{c_{gw}^2 + c_{vx}^2 + c_{vy}^2 + c_{vz}^2}$	$\|\mathbf{c}\|_{\ominus} = \sqrt{c_{vx}^2 + c_{vy}^2 + c_{vz}^2 - c_{gw}^2 + 2\left(c_{gx}c_{mx} + c_{gy}c_{my} + c_{gz}c_{mz}\right)}$
Sphere $\mathbf{s}$	$\|\mathbf{s}\|_{\odot} = \sqrt{s_x^2 + s_y^2 + s_z^2}$	$\|\mathbf{s}\|_{\ominus} = \sqrt{s_x^2 + s_y^2 + s_z^2 - 2s_w s_u}$

GDC

Round Norms

Type	Round Bulk Norm	Round Weight Norm
Round point **a**	$\|\mathbf{a}\|_{\bullet} = \sqrt{a_x^2 + a_y^2 + a_z^2}$	$\|\mathbf{a}\|_{\circ} = |a_w|$
Dipole **d**	$\|\mathbf{d}\|_{\bullet} = \sqrt{d_{mx}^2 + d_{my}^2 + d_{mz}^2}$	$\|\mathbf{d}\|_{\circ} = \sqrt{d_{vx}^2 + d_{vy}^2 + d_{vz}^2}$
Circle **c**	$\|\mathbf{c}\|_{\bullet} = |c_{gw}|$	$\|\mathbf{c}\|_{\circ} = \sqrt{c_{gx}^2 + c_{gy}^2 + c_{gz}^2}$
Sphere **s**	$\|\mathbf{s}\|_{\bullet} = 0$	$\|\mathbf{s}\|_{\circ} = |s_u|$

GDC

Duals

Type	Dual
Flat point $\mathbf{p}$	$\mathbf{p}^* = p_w\,\mathbf{e}_{321} - p_x\,\mathbf{e}_{235} - p_y\,\mathbf{e}_{315} - p_z\,\mathbf{e}_{125}$
Line l	$\mathit{l}^* = l_{vx}\,\mathbf{e}_{23} + l_{vy}\,\mathbf{e}_{31} + l_{vz}\,\mathbf{e}_{12} + l_{mx}\,\mathbf{e}_{15} + l_{my}\,\mathbf{e}_{25} + l_{mz}\,\mathbf{e}_{35}$
Plane $\mathbf{g}$	$\mathbf{g}^* = -g_x\,\mathbf{e}_1 - g_y\,\mathbf{e}_2 - g_z\,\mathbf{e}_3 + g_w\,\mathbf{e}_5$
Round point $\mathbf{a}$	$\mathbf{a}^* = -a_w\,\mathbf{e}_{1234} + a_x\,\mathbf{e}_{4235} + a_y\,\mathbf{e}_{4315} + a_z\,\mathbf{e}_{4125} - a_u\,\mathbf{e}_{3215}$
Dipole $\mathbf{d}$	$\mathbf{d}^* = -d_{vx}\,\mathbf{e}_{423} - d_{vy}\,\mathbf{e}_{431} - d_{vz}\,\mathbf{e}_{412} + d_{pw}\,\mathbf{e}_{321}$ $\quad - d_{mx}\,\mathbf{e}_{415} - d_{my}\,\mathbf{e}_{425} - d_{mz}\,\mathbf{e}_{435} - d_{px}\,\mathbf{e}_{235} - d_{py}\,\mathbf{e}_{315} - d_{pz}\,\mathbf{e}_{125}$
Circle $\mathbf{c}$	$\mathbf{c}^* = c_{gx}\,\mathbf{e}_{41} + c_{gy}\,\mathbf{e}_{42} + c_{gz}\,\mathbf{e}_{43} + c_{vx}\,\mathbf{e}_{23} + c_{vy}\,\mathbf{e}_{31} + c_{vz}\,\mathbf{e}_{12} + c_{mx}\,\mathbf{e}_{15} + c_{my}\,\mathbf{e}_{25} + c_{mz}\,\mathbf{e}_{35} - c_{gw}\,\mathbf{e}_{45}$
Sphere $\mathbf{s}$	$\mathbf{s}^* = -s_x\,\mathbf{e}_1 - s_y\,\mathbf{e}_2 - s_z\,\mathbf{e}_3 + s_u\,\mathbf{e}_4 + s_w\,\mathbf{e}_5$

GDC

Carriers

Carrier is lowest-dimensional flat containing object

Anticarrier is carrier of dual

$$\operatorname{car}(\mathbf{x}) = \mathbf{x} \wedge \mathbf{e}_5$$

$$\operatorname{acr}(\mathbf{x}) = \mathbf{x}^{*} \wedge \mathbf{e}_5$$

Carriers

Type	Carrier	Anticarrier
Round point $\mathbf{a}$	$\mathrm{car}(\mathbf{a}) = a_x\,\mathbf{e}_{15} + a_y\,\mathbf{e}_{25} + a_z\,\mathbf{e}_{35} + a_w\,\mathbf{e}_{45}$	$\mathrm{acr}(\mathbf{a}) = -a_w\mathbb{1}$
Dipole $\mathbf{d}$	$\mathrm{car}(\mathbf{d}) = d_{vx}\,\mathbf{e}_{415} + d_{vy}\,\mathbf{e}_{425} + d_{vz}\,\mathbf{e}_{435}$ $+ d_{mx}\,\mathbf{e}_{235} + d_{my}\,\mathbf{e}_{315} + d_{mz}\,\mathbf{e}_{125}$	$\mathrm{acr}(\mathbf{d}) = -d_{vx}\,\mathbf{e}_{4235} - d_{vy}\,\mathbf{e}_{4315} - d_{vz}\,\mathbf{e}_{4125}$ $+ d_{pw}\,\mathbf{e}_{3215}$
Circle $\mathbf{c}$	$\mathrm{car}(\mathbf{c}) = c_{gx}\,\mathbf{e}_{4235} + c_{gy}\,\mathbf{e}_{4315} + c_{gz}\,\mathbf{e}_{4125}$ $+ c_{gw}\,\mathbf{e}_{3215}$	$\mathrm{acr}(\mathbf{c}) = c_{gx}\,\mathbf{e}_{415} + c_{gy}\,\mathbf{e}_{425} + c_{gz}\,\mathbf{e}_{435}$ $+ c_{vx}\,\mathbf{e}_{235} + c_{vy}\,\mathbf{e}_{315} + c_{vz}\,\mathbf{e}_{125}$
Sphere $\mathbf{s}$	$\mathrm{car}(\mathbf{s}) = s_u\mathbb{1}$	$\mathrm{acr}(\mathbf{s}) = -s_x\,\mathbf{e}_{15} - s_y\,\mathbf{e}_{25} - s_z\,\mathbf{e}_{35} + s_u\,\mathbf{e}_{45}$

Centers

Center is round point with same position and radius

Intersect anticarrier with object itself

$$\operatorname{cen}(\mathbf{x}) = -\operatorname{acr}(\mathbf{x}) \vee \mathbf{x}$$

Centers

Type	Center
Round point $\mathbf{a}$	$\text{cen}(\mathbf{a}) = a_x a_w \mathbf{e}_1 + a_y a_w \mathbf{e}_2 + a_z a_w \mathbf{e}_3 + a_w^2 \mathbf{e}_4 + a_w a_u \mathbf{e}_5$
Dipole $\mathbf{d}$	$\begin{aligned} \text{cen}(\mathbf{d}) = \ & \left(d_{vy} d_{mz} - d_{vz} d_{my} + d_{vx} d_{pw} \right) \mathbf{e}_1 \\ & + \left(d_{vz} d_{mx} - d_{vx} d_{mz} + d_{vy} d_{pw} \right) \mathbf{e}_2 \\ & + \left(d_{vx} d_{my} - d_{vy} d_{mx} + d_{vz} d_{pw} \right) \mathbf{e}_3 \\ & + \left(d_{vx}^2 + d_{vy}^2 + d_{vz}^2 \right) \mathbf{e}_4 + \left(d_{pw}^2 - d_{vx} d_{px} - d_{vy} d_{py} - d_{vz} d_{pz} \right) \mathbf{e}_5 \end{aligned}$
Circle $\mathbf{c}$	$\begin{aligned} \text{cen}(\mathbf{c}) = \ & \left(c_{gy} c_{vz} - c_{gz} c_{vy} - c_{gx} c_{gw} \right) \mathbf{e}_1 \\ & + \left(c_{gz} c_{vx} - c_{gx} c_{vz} - c_{gy} c_{gw} \right) \mathbf{e}_2 \\ & + \left(c_{gx} c_{vy} - c_{gy} c_{vx} - c_{gz} c_{gw} \right) \mathbf{e}_3 \\ & + \left(c_{gx}^2 + c_{gy}^2 + c_{gz}^2 \right) \mathbf{e}_4 + \left(c_{vx}^2 + c_{vy}^2 + c_{vz}^2 + c_{gx} c_{mx} + c_{gy} c_{my} + c_{gz} c_{mz} \right) \mathbf{e}_5 \end{aligned}$
Sphere $\mathbf{s}$	$\text{cen}(\mathbf{s}) = -s_x s_u \mathbf{e}_1 - s_y s_u \mathbf{e}_2 - s_z s_u \mathbf{e}_3 + s_u^2 \mathbf{e}_4 + \left(s_x^2 + s_y^2 + s_z^2 - s_w s_u \right) \mathbf{e}_5$

Containers

Container is sphere with same center and radius

Join dual of carrier with object itself

$$\mathrm{con}\left(\mathbf{x}\right)=\mathrm{car}\left(\mathbf{x}\right)^{*}\wedge\mathbf{x}$$

Containers

Type	Container
Round point $\mathbf{a}$	$\mathrm{con}\left(\mathbf{a}\right) = -a_w^2\,\mathbf{e}_{1234} + a_x a_w\,\mathbf{e}_{4235} + a_y a_w\,\mathbf{e}_{4315} + a_z a_w\,\mathbf{e}_{4125} + \left(a_w a_u - a_x^2 - a_y^2 - a_z^2\right)\mathbf{e}_{3215}$
Dipole $\mathbf{d}$	$\begin{aligned}\mathrm{con}\left(\mathbf{d}\right) =& -\left(d_{vx}^2 + d_{vy}^2 + d_{vz}^2\right)\mathbf{e}_{1234} \\ &+ \left(d_{vy}d_{mz} - d_{vz}d_{my} + d_{vx}d_{pw}\right)\mathbf{e}_{4235} \\ &+ \left(d_{vz}d_{mx} - d_{vx}d_{mz} + d_{vy}d_{pw}\right)\mathbf{e}_{4315} \\ &+ \left(d_{vx}d_{my} - d_{vy}d_{mx} + d_{vz}d_{pw}\right)\mathbf{e}_{4125} \\ &- \left(d_{mx}^2 + d_{my}^2 + d_{mz}^2 + d_{vx}d_{px} + d_{vy}d_{py} + d_{vz}d_{pz}\right)\mathbf{e}_{3215}\end{aligned}$
Circle $\mathbf{c}$	$\begin{aligned}\mathrm{con}\left(\mathbf{c}\right) =& -\left(c_{gx}^2 + c_{gy}^2 + c_{gz}^2\right)\mathbf{e}_{1234} \\ &+ \left(c_{gy}c_{vz} - c_{gz}c_{vy} - c_{gx}c_{gw}\right)\mathbf{e}_{4235} \\ &+ \left(c_{gz}c_{vx} - c_{gx}c_{vz} - c_{gy}c_{gw}\right)\mathbf{e}_{4315} \\ &+ \left(c_{gx}c_{vy} - c_{gy}c_{vx} - c_{gz}c_{gw}\right)\mathbf{e}_{4125} \\ &+ \left(c_{gx}c_{mx} + c_{gy}c_{my} + c_{gz}c_{mz} - c_{gw}^2\right)\mathbf{e}_{3215}\end{aligned}$
Sphere $\mathbf{s}$	$\mathrm{con}\left(\mathbf{s}\right) = -s_u^2\,\mathbf{e}_{1234} - s_x s_u\,\mathbf{e}_{4235} - s_y s_u\,\mathbf{e}_{4315} - s_z s_u\,\mathbf{e}_{4125} - s_w s_u\,\mathbf{e}_{3215}$

GDC

Partners

Partner is same round object with same center, but real or imaginary radius is swapped

Intersect carrier with container of dual

$$\mathrm{par}\left(\mathbf{x}\right) = \mathrm{car}\left(\mathbf{x}\right) \vee \mathrm{con}\left(\mathbf{x}^{*}\right)$$

GDC

Partners

Type	Partner
Round point $\mathbf{a}$	$\operatorname{par}(\mathbf{a}) = a_x a_w^2 \mathbf{e}_1 + a_y a_w^2 \mathbf{e}_2 + a_z a_w^2 \mathbf{e}_3 + a_w^3 \mathbf{e}_4 + \left(a_x^2 + a_y^2 + a_z^2 - a_w a_u \right) a_w \mathbf{e}_5$
Dipole $\mathbf{d}$	$\operatorname{par}(\mathbf{d}) = -\left(d_{vx}^2 + d_{vy}^2 + d_{vz}^2 \right)\left(d_{vx} \mathbf{e}_{41} + d_{vy} \mathbf{e}_{42} + d_{vz} \mathbf{e}_{43} + d_{mx} \mathbf{e}_{23} + d_{my} \mathbf{e}_{31} + d_{mz} \mathbf{e}_{12} + d_{pw} \mathbf{e}_{45} \right)$ $\quad + \left(d_{mx}^2 + d_{my}^2 + d_{mz}^2 - d_{pw}^2 + d_{vx} d_{px} + d_{vy} d_{py} + d_{vz} d_{pz} \right)\left(d_{vx} \mathbf{e}_{15} + d_{vy} \mathbf{e}_{25} + d_{vz} \mathbf{e}_{35} \right)$ $\quad + \left(d_{my} d_{vz} - d_{mz} d_{vy} \right) d_{pw} \mathbf{e}_{15} + \left(d_{mz} d_{vx} - d_{mx} d_{vz} \right) d_{pw} \mathbf{e}_{25} + \left(d_{mx} d_{vy} - d_{my} d_{vx} \right) d_{pw} \mathbf{e}_{35}$
Circle $\mathbf{c}$	$\operatorname{par}(\mathbf{c}) = \quad \left(c_{gx}^2 + c_{gy}^2 + c_{gz}^2 \right)\left(c_{gx} \mathbf{e}_{423} + c_{gy} \mathbf{e}_{431} + c_{gz} \mathbf{e}_{412} + c_{gw} \mathbf{e}_{321} + c_{vx} \mathbf{e}_{415} + c_{vy} \mathbf{e}_{425} + c_{vz} \mathbf{e}_{435} \right)$ $\quad + \left(c_{gw}^2 - c_{vx}^2 - c_{vy}^2 - c_{vz}^2 - c_{gx} c_{mx} - c_{gy} c_{my} - c_{gz} c_{mz} \right)\left(c_{gx} \mathbf{e}_{235} + c_{gy} \mathbf{e}_{315} + c_{gz} \mathbf{e}_{125} \right)$ $\quad + \left(c_{vy} c_{gz} - c_{vz} c_{gy} \right) c_{gw} \mathbf{e}_{235} + \left(c_{vz} c_{gx} - c_{vx} c_{gz} \right) c_{gw} \mathbf{e}_{315} + \left(c_{vx} c_{gy} - c_{vy} c_{gx} \right) c_{gw} \mathbf{e}_{125}$
Sphere $\mathbf{s}$	$\operatorname{par}(\mathbf{s}) = -s_u^3 \mathbf{e}_{1234} - s_x s_u^2 \mathbf{e}_{4235} - s_y s_u^2 \mathbf{e}_{4315} - s_z s_u^2 \mathbf{e}_{4125} + \left(s_w s_u - s_x^2 - s_y^2 - s_z^2 \right) s_u \mathbf{e}_{3215}$

GDC

Attitude

Attitude operation extracts directional information

$$\text{att}\left(\mathbf{x}\right) = \mathbf{x} \vee \overline{\mathbf{e}}_4$$

Attitude

Type	Attitude
Flat point **p**	$\mathrm{att}(\mathbf{p}) = p_w \mathbf{e}_5$
Line l	$\mathrm{att}(l) = l_{vx} \mathbf{e}_{15} + l_{vy} \mathbf{e}_{25} + l_{vz} \mathbf{e}_{35}$
Plane **g**	$\mathrm{att}(\mathbf{g}) = g_x \mathbf{e}_{235} + g_y \mathbf{e}_{315} + g_z \mathbf{e}_{125}$
Round point **a**	$\mathrm{att}(\mathbf{a}) = a_w \mathbf{1}$
Dipole **d**	$\mathrm{att}(\mathbf{d}) = d_{vx} \mathbf{e}_1 + d_{vy} \mathbf{e}_2 + d_{vz} \mathbf{e}_3 + d_{pw} \mathbf{e}_5$
Circle **c**	$\mathrm{att}(\mathbf{c}) = c_{gx} \mathbf{e}_{23} + c_{gy} \mathbf{e}_{31} + c_{gz} \mathbf{e}_{12} + c_{vx} \mathbf{e}_{15} + c_{vy} \mathbf{e}_{25} + c_{vz} \mathbf{e}_{35}$
Sphere **s**	$\mathrm{att}(\mathbf{s}) = s_u \mathbf{e}_{321} + s_x \mathbf{e}_{235} + s_y \mathbf{e}_{315} + s_z \mathbf{e}_{125}$

Join

| Formula	Illustration
Dipole containing round points **a** and **b**. $\mathbf{a} \wedge \mathbf{b} = (a_w b_x - a_x b_w)\,\mathbf{e}_{41} + (a_w b_y - a_y b_w)\,\mathbf{e}_{42} + (a_w b_z - a_z b_w)\,\mathbf{e}_{43}$ $\qquad + (a_y b_z - a_z b_y)\,\mathbf{e}_{23} + (a_z b_x - a_x b_z)\,\mathbf{e}_{31} + (a_x b_y - a_y b_x)\,\mathbf{e}_{12}$ $\qquad + (a_x b_u - a_u b_x)\,\mathbf{e}_{15} + (a_y b_u - a_u b_y)\,\mathbf{e}_{25}$ $\qquad + (a_z b_u - a_u b_z)\,\mathbf{e}_{35} + (a_w b_u - a_u b_w)\,\mathbf{e}_{45}$	
Line containing flat point **p** and round point **a**. $\mathbf{p} \wedge \mathbf{a} = (p_x a_w - p_w a_x)\,\mathbf{e}_{415} + (p_z a_y - p_y a_z)\,\mathbf{e}_{235}$ $\qquad + (p_y a_w - p_w a_y)\,\mathbf{e}_{425} + (p_x a_z - p_z a_x)\,\mathbf{e}_{315}$ $\qquad + (p_z a_w - p_w a_z)\,\mathbf{e}_{435} + (p_y a_x - p_x a_y)\,\mathbf{e}_{125}$	
Circle containing dipole **d** and round point **a**. $\mathbf{d} \wedge \mathbf{a} = (d_{vy} a_z - d_{vz} a_y + d_{mx} a_w)\,\mathbf{e}_{423} + (d_{vz} a_x - d_{vx} a_z + d_{my} a_w)\,\mathbf{e}_{431}$ $\qquad + (d_{vx} a_y - d_{vy} a_x + d_{mz} a_w)\,\mathbf{e}_{412} - (d_{mx} a_x + d_{my} a_y + d_{mz} a_z)\,\mathbf{e}_{321}$ $\qquad + (d_{px} a_w - d_{pw} a_x + d_{vx} a_u)\,\mathbf{e}_{415} + (d_{pz} a_y - d_{py} a_z + d_{mx} a_u)\,\mathbf{e}_{235}$ $\qquad + (d_{py} a_w - d_{pw} a_y + d_{vy} a_u)\,\mathbf{e}_{425} + (d_{px} a_z - d_{pz} a_x + d_{my} a_u)\,\mathbf{e}_{315}$ $\qquad + (d_{pz} a_w - d_{pw} a_z + d_{vz} a_u)\,\mathbf{e}_{435} + (d_{py} a_x - d_{px} a_y + d_{mz} a_u)\,\mathbf{e}_{125}$	
Plane containing line *l* and round point **a**. $l \wedge \mathbf{a} = (l_{vz} a_y - l_{vy} a_z - l_{mx} a_w)\,\mathbf{e}_{4235} + (l_{vx} a_z - l_{vz} a_x - l_{my} a_w)\,\mathbf{e}_{4315}$ $\qquad + (l_{vy} a_x - l_{vx} a_y - l_{mz} a_w)\,\mathbf{e}_{4125} + (l_{mx} a_x + l_{my} a_y + l_{mz} a_z)\,\mathbf{e}_{3215}$	
Plane containing dipole **d** and flat point **p**. $\mathbf{d} \wedge \mathbf{p} = (d_{vy} p_z - d_{vz} p_y + d_{mx} p_w)\,\mathbf{e}_{4235}$ $\qquad + (d_{vz} p_x - d_{vx} p_z + d_{my} p_w)\,\mathbf{e}_{4315}$ $\qquad + (d_{vx} p_y - d_{vy} p_x + d_{mz} p_w)\,\mathbf{e}_{4125}$ $\qquad - (d_{mx} p_x + d_{my} p_y + d_{mz} p_z)\,\mathbf{e}_{3215}$	
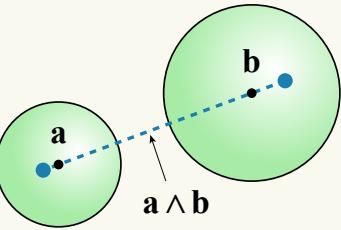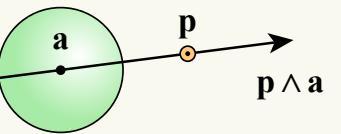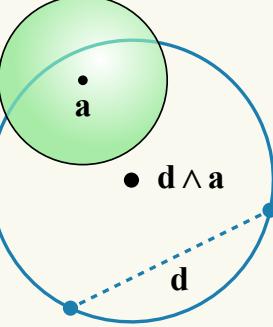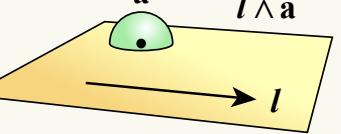
GDC

Join

Formula	Illustration
Sphere containing circle **c** and round point **a**. $\mathbf{c} \wedge \mathbf{a} = -\left(c_{gx} a_x + c_{gy} a_y + c_{gz} a_z + c_{gw} a_w \right) \mathbf{e}_{1234}$ $\qquad + \left(c_{vz} a_y - c_{vy} a_z + c_{gx} a_u - c_{mx} a_w \right) \mathbf{e}_{4235}$ $\qquad + \left(c_{vx} a_z - c_{vz} a_x + c_{gy} a_u - c_{my} a_w \right) \mathbf{e}_{4315}$ $\qquad + \left(c_{vy} a_x - c_{vx} a_y + c_{gz} a_u - c_{mz} a_w \right) \mathbf{e}_{4125}$ $\qquad + \left(c_{mx} a_x + c_{my} a_y + c_{mz} a_z + c_{gw} a_u \right) \mathbf{e}_{3215}$	
Sphere containing dipoles **d** and **f**. $\mathbf{d} \wedge \mathbf{f} = -\left(d_{vx} f_{mx} + d_{vy} f_{my} + d_{vz} f_{mz} + d_{mx} f_{vx} + d_{my} f_{vy} + d_{mz} f_{vz} \right) \mathbf{e}_{1234}$ $\qquad + \left(d_{vy} f_{pz} - d_{vz} f_{py} + d_{pz} f_{vy} - d_{py} f_{vz} + d_{mx} f_{pw} + d_{pw} f_{mx} \right) \mathbf{e}_{4235}$ $\qquad + \left(d_{vz} f_{px} - d_{vx} f_{pz} + d_{px} f_{vz} - d_{pz} f_{vx} + d_{my} f_{pw} + d_{pw} f_{my} \right) \mathbf{e}_{4315}$ $\qquad + \left(d_{vx} f_{py} - d_{vy} p_{px} + d_{py} f_{vx} - d_{px} f_{vy} + d_{mz} f_{pw} + d_{pw} f_{mz} \right) \mathbf{e}_{4125}$ $\qquad - \left(d_{mx} f_{px} + d_{my} f_{py} + d_{mz} f_{pz} + d_{px} f_{mx} + d_{py} f_{my} + d_{pz} f_{mz} \right) \mathbf{e}_{3215}$	
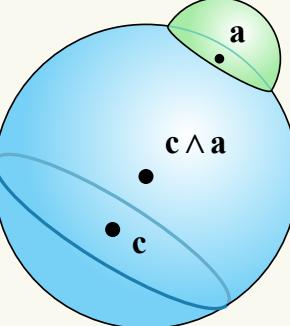
GDC

Meet

Formula	Illustration
Circle where spheres $\mathbf{s}$ and $\mathbf{t}$ intersect. $\mathbf{s} \vee \mathbf{t} = \left(s_u t_x - s_x t_u \right) \mathbf{e}_{423} + \left(s_u t_y - s_y t_u \right) \mathbf{e}_{431}$ $\quad + \left(s_u t_z - s_z t_u \right) \mathbf{e}_{412} + \left(s_u t_w - s_w t_u \right) \mathbf{e}_{321}$ $\quad + \left(s_z t_y - s_y t_z \right) \mathbf{e}_{415} + \left(s_x t_z - s_z t_x \right) \mathbf{e}_{425} + \left(s_y t_x - s_x t_y \right) \mathbf{e}_{435}$ $\quad + \left(s_x t_w - s_w t_x \right) \mathbf{e}_{235} + \left(s_y t_w - s_w t_y \right) \mathbf{e}_{315} + \left(s_z t_w - s_w t_z \right) \mathbf{e}_{125}$	
Circle where sphere $\mathbf{s}$ and plane $\mathbf{g}$ intersect. $\mathbf{s} \vee \mathbf{g} = s_u g_x \mathbf{e}_{423} + s_u g_y \mathbf{e}_{431} + s_u g_z \mathbf{e}_{412} + s_u g_w \mathbf{e}_{321}$ $\quad + \left(s_z g_y - s_y g_z \right) \mathbf{e}_{415} + \left(s_x g_z - s_z g_x \right) \mathbf{e}_{425} + \left(s_y g_x - s_x g_y \right) \mathbf{e}_{435}$ $\quad + \left(s_x g_w - s_w g_x \right) \mathbf{e}_{235} + \left(s_y g_w - s_w g_y \right) \mathbf{e}_{315} + \left(s_z g_w - s_w g_z \right) \mathbf{e}_{125}$	
Line where planes $\mathbf{g}$ and $\mathbf{h}$ intersect. $\mathbf{g} \vee \mathbf{h} = \left(g_z h_y - g_y h_z \right) \mathbf{e}_{415} + \left(g_x h_w - g_w h_x \right) \mathbf{e}_{235}$ $\quad + \left(g_x h_z - g_z h_x \right) \mathbf{e}_{425} + \left(g_y h_w - g_w h_y \right) \mathbf{e}_{315}$ $\quad + \left(g_y h_x - g_x h_y \right) \mathbf{e}_{435} + \left(g_z h_w - g_w h_z \right) \mathbf{e}_{125}$	
Dipole where sphere $\mathbf{s}$ and circle $\mathbf{c}$ intersect. $\mathbf{s} \vee \mathbf{c} = \left(s_y c_{gz} - s_z c_{gy} + s_u c_{vx} \right) \mathbf{e}_{41} + \left(s_w c_{gx} - s_x c_{gw} + s_u c_{mx} \right) \mathbf{e}_{23}$ $\quad + \left(s_z c_{gx} - s_x c_{gz} + s_u c_{vy} \right) \mathbf{e}_{42} + \left(s_w c_{gy} - s_y c_{gw} + s_u c_{my} \right) \mathbf{e}_{31}$ $\quad + \left(s_x c_{gy} - s_y c_{gx} + s_u c_{vz} \right) \mathbf{e}_{43} + \left(s_w c_{gz} - s_z c_{gw} + s_u c_{mz} \right) \mathbf{e}_{12}$ $\quad + \left(s_z c_{my} - s_y c_{mz} + s_w c_{vx} \right) \mathbf{e}_{15} + \left(s_x c_{mz} - s_z c_{mx} + s_w c_{vy} \right) \mathbf{e}_{25}$ $\quad + \left(s_y c_{mx} - s_x c_{my} + s_w c_{vz} \right) \mathbf{e}_{35} - \left(s_x c_{vx} + s_y c_{vy} + s_z c_{vz} \right) \mathbf{e}_{45}$	
Dipole where plane $\mathbf{g}$ and circle $\mathbf{c}$ intersect. $\mathbf{g} \vee \mathbf{c} = \left(g_y c_{gz} - g_z c_{gy} \right) \mathbf{e}_{41} + \left(g_w c_{gx} - g_x c_{gw} \right) \mathbf{e}_{23}$ $\quad + \left(g_z c_{gx} - g_x c_{gz} \right) \mathbf{e}_{42} + \left(g_w c_{gy} - g_y c_{gw} \right) \mathbf{e}_{31}$ $\quad + \left(g_x c_{gy} - g_y c_{gx} \right) \mathbf{e}_{43} + \left(g_w c_{gz} - g_z c_{gw} \right) \mathbf{e}_{12}$ $\quad + \left(g_z c_{my} - g_y c_{mz} + g_w c_{vx} \right) \mathbf{e}_{15} + \left(g_x c_{mz} - g_z c_{mx} + g_w c_{vy} \right) \mathbf{e}_{25}$ $\quad + \left(g_y c_{mx} - g_x c_{my} + g_w c_{vz} \right) \mathbf{e}_{35} - \left(g_x c_{vx} + g_y c_{vy} + g_z c_{vz} \right) \mathbf{e}_{45}$	
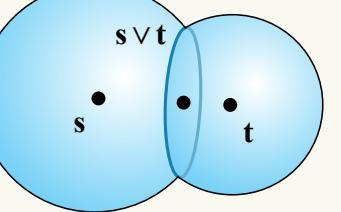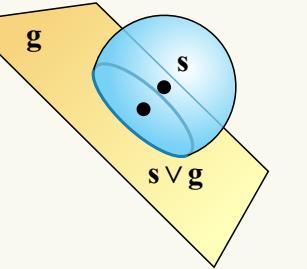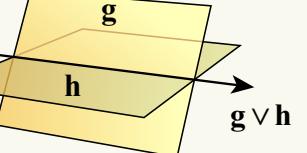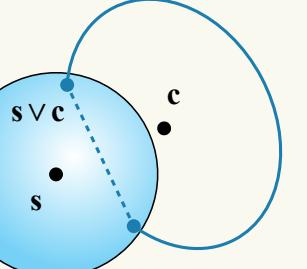
GDC

Meet

Formula	Illustration
Dipole where sphere $\mathbf{s}$ and line l intersect. $\begin{aligned}\mathbf{s} \vee l =\ & s_u l_{vx}\mathbf{e}_{41} + s_u l_{vy}\mathbf{e}_{42} + s_u l_{vz}\mathbf{e}_{43}\\ & + s_u l_{mx}\mathbf{e}_{23} + s_u l_{my}\mathbf{e}_{31} + s_u l_{mz}\mathbf{e}_{12}\\ & + \left(s_z l_{my} - s_y l_{mz} + s_w l_{vx}\right)\mathbf{e}_{15} + \left(s_x l_{mz} - s_z l_{mx} + s_w l_{vy}\right)\mathbf{e}_{25}\\ & + \left(s_y l_{mx} - s_x l_{my} + s_w l_{vz}\right)\mathbf{e}_{35} - \left(s_x l_{vx} + s_y l_{vy} + s_z l_{vz}\right)\mathbf{e}_{45}\end{aligned}$	
Flat point where plane $\mathbf{g}$ and line l intersect. $\begin{aligned}\mathbf{g} \vee l =\ & \left(g_z l_{my} - g_y l_{mz} + g_w l_{vx}\right)\mathbf{e}_{15} + \left(g_x l_{mz} - g_z l_{mx} + g_w l_{vy}\right)\mathbf{e}_{25}\\ & + \left(g_y l_{mx} - g_x l_{my} + g_w l_{vz}\right)\mathbf{e}_{35} - \left(g_x l_{vx} + g_y l_{vy} + g_z l_{vz}\right)\mathbf{e}_{45}\end{aligned}$	
Round point contained by circles $\mathbf{c}$ and $\mathbf{o}$. $\begin{aligned}\mathbf{c} \vee \mathbf{o} =\ & \left(c_{gz}o_{my} - c_{gy}o_{mz} + c_{my}o_{gz} - c_{mz}o_{gy} + c_{vx}o_{gw} + c_{gw}o_{vx}\right)\mathbf{e}_1\\ & + \left(c_{gx}o_{mz} - c_{gz}o_{mx} + c_{mz}o_{gx} - c_{mx}o_{gz} + c_{vy}o_{gw} + c_{gw}o_{vy}\right)\mathbf{e}_2\\ & + \left(c_{gy}o_{mx} - c_{gx}o_{my} + c_{mx}o_{gy} - c_{my}o_{gx} + c_{vz}o_{gw} + c_{gw}o_{vz}\right)\mathbf{e}_3\\ & - \left(c_{gx}o_{vx} + c_{gy}o_{vy} + c_{gz}o_{vz} + c_{vx}o_{gx} + c_{vy}o_{gy} + c_{vz}o_{gz}\right)\mathbf{e}_4\\ & - \left(c_{mx}o_{vx} + c_{my}o_{vy} + c_{mz}o_{vz} + c_{vx}o_{mx} + c_{vy}o_{my} + c_{vz}o_{mz}\right)\mathbf{e}_5\end{aligned}$	
Round point centered on line l and contained by circle $\mathbf{c}$. $\begin{aligned}\mathbf{c} \vee l =\ & \left(c_{gz}l_{my} - c_{gy}l_{mz} + c_{gw}l_{vx}\right)\mathbf{e}_1 + \left(c_{gx}l_{mz} - c_{gz}l_{mx} + c_{gw}l_{vy}\right)\mathbf{e}_2\\ & + \left(c_{gy}l_{mx} - c_{gx}l_{my} + c_{gw}l_{vz}\right)\mathbf{e}_3 - \left(c_{gx}l_{vx} + c_{gy}l_{vy} + c_{vz}l_{vz}\right)\mathbf{e}_4\\ & - \left(c_{mx}l_{vx} + c_{my}l_{vy} + c_{mz}l_{vz} + c_{vx}l_{mx} + c_{vy}l_{my} + c_{vz}l_{mz}\right)\mathbf{e}_5\end{aligned}$	
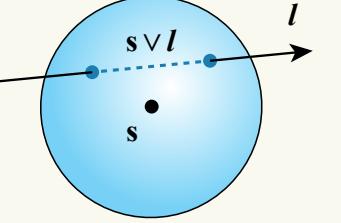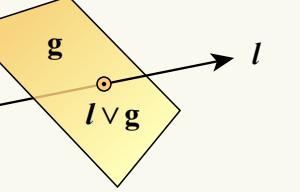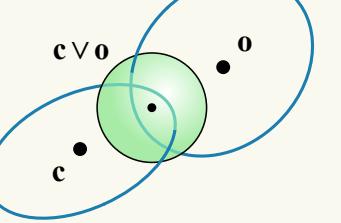
Meet

Formula	Illustration
Round point contained by sphere **s** and dipole **d**. $\mathbf{s} \vee \mathbf{d} = \left(s_y d_{mz} - s_z d_{my} - s_w d_{vx} + s_u d_{px} \right) \mathbf{e}_1$ $\qquad + \left(s_z d_{mx} - s_x d_{mz} - s_w d_{vy} + s_u d_{py} \right) \mathbf{e}_2$ $\qquad + \left(s_x d_{my} - s_y d_{mx} - s_w d_{vz} + s_u d_{pz} \right) \mathbf{e}_3$ $\qquad + \left(s_x d_{vx} + s_y d_{vy} + s_z d_{vz} + s_u d_{pw} \right) \mathbf{e}_4$ $\qquad - \left(s_x d_{px} + s_y d_{py} + s_z d_{pz} + s_w d_{pw} \right) \mathbf{e}_5$	
Round point centered in plane **g** and contained by dipole **d**. $\mathbf{g} \vee \mathbf{d} = \left(g_y d_{mz} - g_z d_{my} - g_w d_{vx} \right) \mathbf{e}_1$ $\qquad + \left(g_z d_{mx} - g_x d_{mz} - g_w d_{vy} \right) \mathbf{e}_2$ $\qquad + \left(g_x d_{my} - g_y d_{mx} - g_w d_{vz} \right) \mathbf{e}_3$ $\qquad + \left(g_x d_{vx} + g_y d_{vy} + g_z d_{vz} \right) \mathbf{e}_4$ $\qquad - \left(g_x d_{px} + g_y d_{py} + g_z d_{pz} + g_w d_{pw} \right) \mathbf{e}_5$	
Round point centered at flat point **p** and contained by sphere **s**. $\mathbf{s} \vee \mathbf{p} = s_u p_x \mathbf{e}_1 + s_u p_y \mathbf{e}_2 + s_u p_z \mathbf{e}_3 + s_u p_w \mathbf{e}_4$ $\qquad - \left(s_x p_x + s_y p_y + s_z p_z + s_w p_w \right) \mathbf{e}_5$	
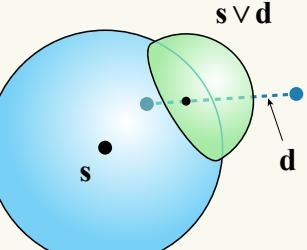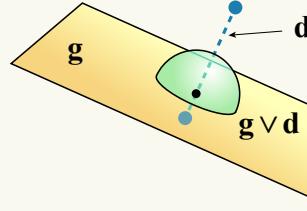
Connect

Formula	Illustration
Dipole orthogonal to sphere $\mathbf{s}$ and containing round point $\mathbf{a}$. $\mathbf{s}^* \wedge \mathbf{a} = (s_x a_w + s_u a_x)\,\mathbf{e}_{41} + (s_z a_y - s_y a_z)\,\mathbf{e}_{23}$ $\qquad + (s_y a_w + s_u a_y)\,\mathbf{e}_{42} + (s_x a_z - s_z a_x)\,\mathbf{e}_{31}$ $\qquad + (s_z a_w + s_u a_z)\,\mathbf{e}_{43} + (s_y a_x - s_x a_y)\,\mathbf{e}_{12}$ $\qquad - (s_x a_u + s_w a_x)\,\mathbf{e}_{15} - (s_y a_u + s_w a_y)\,\mathbf{e}_{25}$ $\qquad - (s_z a_u + s_w a_z)\,\mathbf{e}_{35} + (s_u a_u - s_w a_w)\,\mathbf{e}_{45}$	
Dipole orthogonal to plane $\mathbf{g}$ and containing round point $\mathbf{a}$. $\mathbf{g}^* \wedge \mathbf{a} = g_x a_w \mathbf{e}_{41} + (g_z a_y - g_y a_z)\,\mathbf{e}_{23}$ $\qquad + g_y a_w \mathbf{e}_{42} + (g_x a_z - g_z a_x)\,\mathbf{e}_{31}$ $\qquad + g_z a_w \mathbf{e}_{43} + (g_y a_x - g_x a_y)\,\mathbf{e}_{12}$ $\qquad - (g_x a_u + g_w a_x)\,\mathbf{e}_{15} - (g_y a_u + g_w a_y)\,\mathbf{e}_{25}$ $\qquad - (g_z a_u + g_w a_z)\,\mathbf{e}_{35} - g_w a_w \mathbf{e}_{45}$	
Circle orthogonal to sphere $\mathbf{s}$ and containing dipole $\mathbf{d}$. $\mathbf{s}^* \wedge \mathbf{d} = (s_y d_{vz} - s_z d_{vy} + s_u d_{mx})\,\mathbf{e}_{423} + (s_z d_{vx} - s_x d_{vz} + s_u d_{my})\,\mathbf{e}_{431}$ $\qquad + (s_x d_{vy} - s_y d_{vx} + s_u d_{mz})\,\mathbf{e}_{412} + (s_x d_{mx} + s_y d_{my} + s_z d_{mz})\,\mathbf{e}_{321}$ $\qquad + (s_x d_{pw} + s_w d_{vx} + s_u d_{px})\,\mathbf{e}_{415} + (s_z d_{py} - s_y d_{pz} + s_w d_{mx})\,\mathbf{e}_{235}$ $\qquad + (s_y d_{pw} + s_w d_{vy} + s_u d_{py})\,\mathbf{e}_{425} + (s_x d_{pz} - s_z d_{px} + s_w d_{my})\,\mathbf{e}_{315}$ $\qquad + (s_z d_{pw} + s_w d_{vz} + s_u d_{pz})\,\mathbf{e}_{435} + (s_y d_{px} - s_x d_{py} + s_w d_{mz})\,\mathbf{e}_{125}$	
Circle orthogonal to plane $\mathbf{g}$ and containing dipole $\mathbf{d}$. $\mathbf{g}^* \wedge \mathbf{d} = (g_y d_{vz} - g_z d_{vy})\,\mathbf{e}_{423} + (g_z d_{vx} - g_x d_{vz})\,\mathbf{e}_{431}$ $\qquad + (g_x d_{vy} - g_y d_{vx})\,\mathbf{e}_{412} + (g_x d_{mx} + g_y d_{my} + g_z d_{mz})\,\mathbf{e}_{321}$ $\qquad + (g_x d_{pw} + g_w d_{vx})\,\mathbf{e}_{415} + (g_z d_{py} - g_y d_{pz} + g_w d_{mx})\,\mathbf{e}_{235}$ $\qquad + (g_y d_{pw} + g_w d_{vy})\,\mathbf{e}_{425} + (g_x d_{pz} - g_z d_{px} + g_w d_{my})\,\mathbf{e}_{315}$ $\qquad + (g_z d_{pw} + g_w d_{vz})\,\mathbf{e}_{435} + (g_y d_{px} - g_x d_{py} + g_w d_{mz})\,\mathbf{e}_{125}$	

GDC

Connect

Formula	Illustration
Line orthogonal to sphere $\mathbf{s}$ and containing flat point $\mathbf{p}$. $\mathbf{s}^* \wedge \mathbf{p} = (s_x p_w + s_u p_x)\mathbf{e}_{415} + (s_z p_y - s_y p_z)\mathbf{e}_{235}$ $\qquad + (s_y p_w + s_u p_y)\mathbf{e}_{425} + (s_x p_z - s_z p_x)\mathbf{e}_{315}$ $\qquad + (s_z p_w + s_u p_z)\mathbf{e}_{435} + (s_y p_x - s_x p_y)\mathbf{e}_{125}$	
Line orthogonal to plane $\mathbf{g}$ and containing flat point $\mathbf{p}$. $\mathbf{g}^* \wedge \mathbf{p} = g_x p_w \mathbf{e}_{415} + (g_z p_y - g_y p_z)\mathbf{e}_{235}$ $\qquad + g_y p_w \mathbf{e}_{425} + (g_x p_z - g_z p_x)\mathbf{e}_{315}$ $\qquad + g_z p_w \mathbf{e}_{435} + (g_y p_x - g_x p_y)\mathbf{e}_{125}$	
Sphere orthogonal to sphere $\mathbf{s}$ and containing circle $\mathbf{c}$. $\mathbf{s}^* \wedge \mathbf{c} = (s_u c_{gw} - s_x c_{gx} - s_y c_{gy} - s_z c_{gz})\mathbf{e}_{1234}$ $\qquad + (s_y c_{vz} - s_z c_{vy} - s_w c_{gx} + s_u c_{mx})\mathbf{e}_{4235}$ $\qquad + (s_z c_{vx} - s_x c_{vz} - s_w c_{gy} + s_u c_{my})\mathbf{e}_{4315}$ $\qquad + (s_x c_{vy} - s_y c_{vx} - s_w c_{gz} + s_u c_{mz})\mathbf{e}_{4125}$ $\qquad + (s_x c_{mx} + s_y c_{my} + s_z c_{mz} - s_w c_{gw})\mathbf{e}_{3215}$	
Sphere orthogonal to plane $\mathbf{g}$ and containing circle $\mathbf{c}$. $\mathbf{g}^* \wedge \mathbf{c} = -(g_x c_{gx} + g_y c_{gy} + g_z c_{gz})\mathbf{e}_{1234}$ $\qquad + (g_y c_{vz} - g_z c_{vy} - g_w c_{gx})\mathbf{e}_{4235}$ $\qquad + (g_z c_{vx} - g_x c_{vz} - g_w c_{gy})\mathbf{e}_{4315}$ $\qquad + (g_x c_{vy} - g_y c_{vx} - g_w c_{gz})\mathbf{e}_{4125}$ $\qquad + (g_x c_{mx} + g_y c_{my} + g_z c_{mz} - g_w c_{gw})\mathbf{e}_{3215}$	
Plane orthogonal to sphere $\mathbf{s}$ and containing line $\boldsymbol{l}$. $\mathbf{s}^* \wedge \boldsymbol{l} = (s_y l_{vz} - s_z l_{vy} + s_u l_{mx})\mathbf{e}_{4235} + (s_z l_{vx} - s_x l_{vz} + s_u l_{my})\mathbf{e}_{4315}$ $\qquad + (s_x l_{vy} - s_y l_{vx} + s_u l_{mz})\mathbf{e}_{4125} + (s_x l_{mx} + s_y l_{my} + s_z l_{mz})\mathbf{e}_{3215}$	
Plane orthogonal to plane $\mathbf{g}$ and containing line $\boldsymbol{l}$. $\mathbf{g}^* \wedge \boldsymbol{l} = (g_y l_{vz} - g_z l_{vy})\mathbf{e}_{4235} + (g_z l_{vx} - g_x l_{vz})\mathbf{e}_{4315}$ $\qquad + (g_x l_{vy} - g_y l_{vx})\mathbf{e}_{4125} + (g_x l_{mx} + g_y l_{my} + g_z l_{mz})\mathbf{e}_{3215}$	

GDC

Connect

Formula	Illustration
Circle orthogonal to circle $\mathbf{c}$ and containing round point $\mathbf{a}$. $$\begin{aligned} \mathbf{c}^* \wedge \mathbf{a} = {}& \left(c_{gy}a_z - c_{gz}a_y + c_{vx}a_w \right) \mathbf{e}_{423} + \left(c_{gz}a_x - c_{gx}a_z + c_{vy}a_w \right) \mathbf{e}_{431} \\ &+ \left(c_{gx}a_y - c_{gy}a_x + c_{vz}a_w \right) \mathbf{e}_{412} - \left(c_{vx}a_x + c_{vy}a_y + c_{vz}a_z \right) \mathbf{e}_{321} \\ &+ \left(c_{mx}a_w + c_{gw}a_x + c_{gx}a_u \right) \mathbf{e}_{415} + \left(c_{mz}a_y - c_{my}a_z + c_{vx}a_u \right) \mathbf{e}_{235} \\ &+ \left(c_{my}a_w + c_{gw}a_y + c_{gy}a_u \right) \mathbf{e}_{425} + \left(c_{mx}a_z - c_{mz}a_x + c_{vy}a_u \right) \mathbf{e}_{315} \\ &+ \left(c_{mz}a_w + c_{gw}a_z + c_{gz}a_u \right) \mathbf{e}_{435} + \left(c_{my}a_x - c_{mx}a_y + c_{vz}a_u \right) \mathbf{e}_{125} \end{aligned}$$	
Circle orthogonal to line l and containing round point $\mathbf{a}$. $$\begin{aligned} l^* \wedge \mathbf{a} = {}& l_{vx}a_w \mathbf{e}_{423} + l_{vy}a_w \mathbf{e}_{431} + l_{vz}a_w \mathbf{e}_{412} \\ &- \left(l_{vx}a_x + l_{vy}a_y + l_{vz}a_z \right) \mathbf{e}_{321} \\ &+ l_{mx}a_w \mathbf{e}_{415} + \left(l_{mz}a_y - l_{my}a_z + l_{vx}a_u \right) \mathbf{e}_{235} \\ &+ l_{my}a_w \mathbf{e}_{425} + \left(l_{mx}a_z - l_{mz}a_x + l_{vy}a_u \right) \mathbf{e}_{315} \\ &+ l_{mz}a_w \mathbf{e}_{435} + \left(l_{my}a_x - l_{mx}a_y + l_{vz}a_u \right) \mathbf{e}_{125} \end{aligned}$$	
Plane orthogonal to circle $\mathbf{c}$ and containing flat point $\mathbf{p}$. $$\begin{aligned} \mathbf{c}^* \wedge \mathbf{p} = {}& \left(c_{gy}p_z - c_{gz}p_y + c_{vx}p_w \right) \mathbf{e}_{4235} \\ &+ \left(c_{gz}p_x - c_{gx}p_z + c_{vy}p_w \right) \mathbf{e}_{4315} \\ &+ \left(c_{gx}p_y - c_{gy}p_x + c_{vz}p_w \right) \mathbf{e}_{4125} \\ &- \left(c_{vx}p_x + c_{vy}p_y + c_{vz}p_z \right) \mathbf{e}_{3215} \end{aligned}$$	
Plane orthogonal to line l and containing flat point $\mathbf{p}$. $$\begin{aligned} l^* \wedge \mathbf{p} = {}& l_{vx}p_w \mathbf{e}_{4235} + l_{vy}p_w \mathbf{e}_{4315} + l_{vz}p_w \mathbf{e}_{4125} \\ &- \left(l_{vx}p_x + l_{vy}p_y + l_{vz}p_z \right) \mathbf{e}_{3215} \end{aligned}$$	

GDC

Connect

| Formula	Illustration
Sphere orthogonal to circle **c** and containing dipole **d**. $\mathbf{c}^* \wedge \mathbf{d} = -\left(c_{vx}d_{vx} + c_{vy}d_{vy} + c_{vz}d_{vz} + c_{gx}d_{mx} + c_{gy}d_{my} + c_{gz}d_{mz} \right)\mathbf{e}_{1234}$ $\quad + \left(c_{mz}d_{vy} - c_{my}d_{vz} + c_{vx}d_{pw} + c_{gy}d_{pz} - c_{gz}d_{py} - c_{gw}d_{mx} \right)\mathbf{e}_{4235}$ $\quad + \left(c_{mx}d_{vz} - c_{mz}d_{vx} + c_{vy}d_{pw} + c_{gz}d_{px} - c_{gx}d_{pz} - c_{gw}d_{my} \right)\mathbf{e}_{4315}$ $\quad + \left(c_{my}d_{vx} - c_{mx}d_{vy} + c_{vz}d_{pw} + c_{gx}d_{py} - c_{gy}d_{px} - c_{gw}d_{mz} \right)\mathbf{e}_{4125}$ $\quad - \left(c_{vx}d_{px} + c_{vy}d_{py} + c_{vz}d_{pz} + c_{mx}d_{mx} + c_{my}d_{my} + c_{mz}d_{mz} \right)\mathbf{e}_{3215}$	
Sphere orthogonal to line *l* and containing dipole **d**. $\mathbf{l}^* \wedge \mathbf{d} = -\left(l_{vx}d_{vx} + l_{vy}d_{vy} + l_{vz}d_{vz} \right)\mathbf{e}_{1234}$ $\quad + \left(l_{mz}d_{vy} - l_{my}d_{vz} + l_{vx}d_{pw} \right)\mathbf{e}_{4235}$ $\quad + \left(l_{mx}d_{vz} - l_{mz}d_{vx} + l_{vy}d_{pw} \right)\mathbf{e}_{4315}$ $\quad + \left(l_{my}d_{vx} - l_{mx}d_{vy} + l_{vz}d_{pw} \right)\mathbf{e}_{4125}$ $\quad - \left(l_{vx}d_{px} + l_{vy}d_{py} + l_{vz}d_{pz} + l_{mx}d_{mx} + l_{my}d_{my} + l_{mz}d_{mz} \right)\mathbf{e}_{3215}$	
Sphere orthogonal to dipole **d** and containing round point **a**. $\mathbf{d}^* \wedge \mathbf{a} = \left(d_{vx}a_x + d_{vy}a_y + d_{vz}a_z - d_{pw}a_w \right)\mathbf{e}_{1234}$ $\quad + \left(d_{my}a_z - d_{mz}a_y + d_{px}a_w - d_{vx}a_u \right)\mathbf{e}_{4235}$ $\quad + \left(d_{mz}a_x - d_{mx}a_z + d_{py}a_w - d_{vy}a_u \right)\mathbf{e}_{4315}$ $\quad + \left(d_{mx}a_y - d_{my}a_x + d_{pz}a_w - d_{vz}a_u \right)\mathbf{e}_{4125}$ $\quad + \left(d_{pw}a_u - d_{px}a_x - d_{py}a_y - d_{pz}a_z \right)\mathbf{e}_{3215}$	
Sphere centered at flat point **p** and containing round point **a**. $\mathbf{p}^* \wedge \mathbf{a} = -p_w a_w \mathbf{e}_{1234} + p_x a_w \mathbf{e}_{4235} + p_y a_w \mathbf{e}_{4315} + p_z a_w \mathbf{e}_{4125}$ $\quad + \left(p_w a_u - p_x a_x - p_y a_y - p_z a_z \right)\mathbf{e}_{3215}$	

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Projections

Same as rigid geometric algebra

To project $\mathbf{a}$ onto $\mathbf{b}$, calculate connect $\mathbf{b}^* \wedge \mathbf{a}$

Then intersect result with $\mathbf{b}$: $\left(\mathbf{b}^* \wedge \mathbf{a} \right) \vee \mathbf{b}$

Points Contained by Sphere
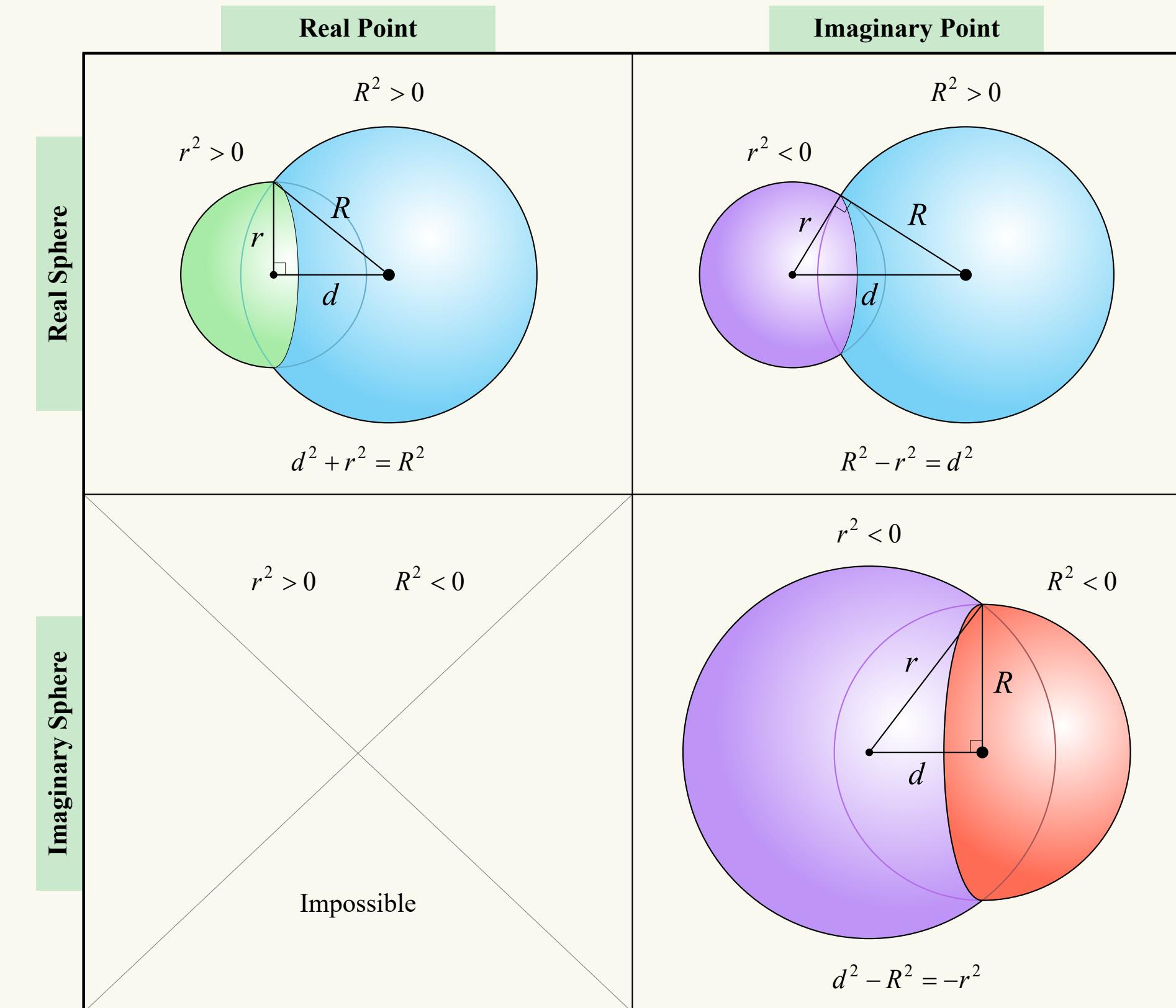
Points Contained by Sphere
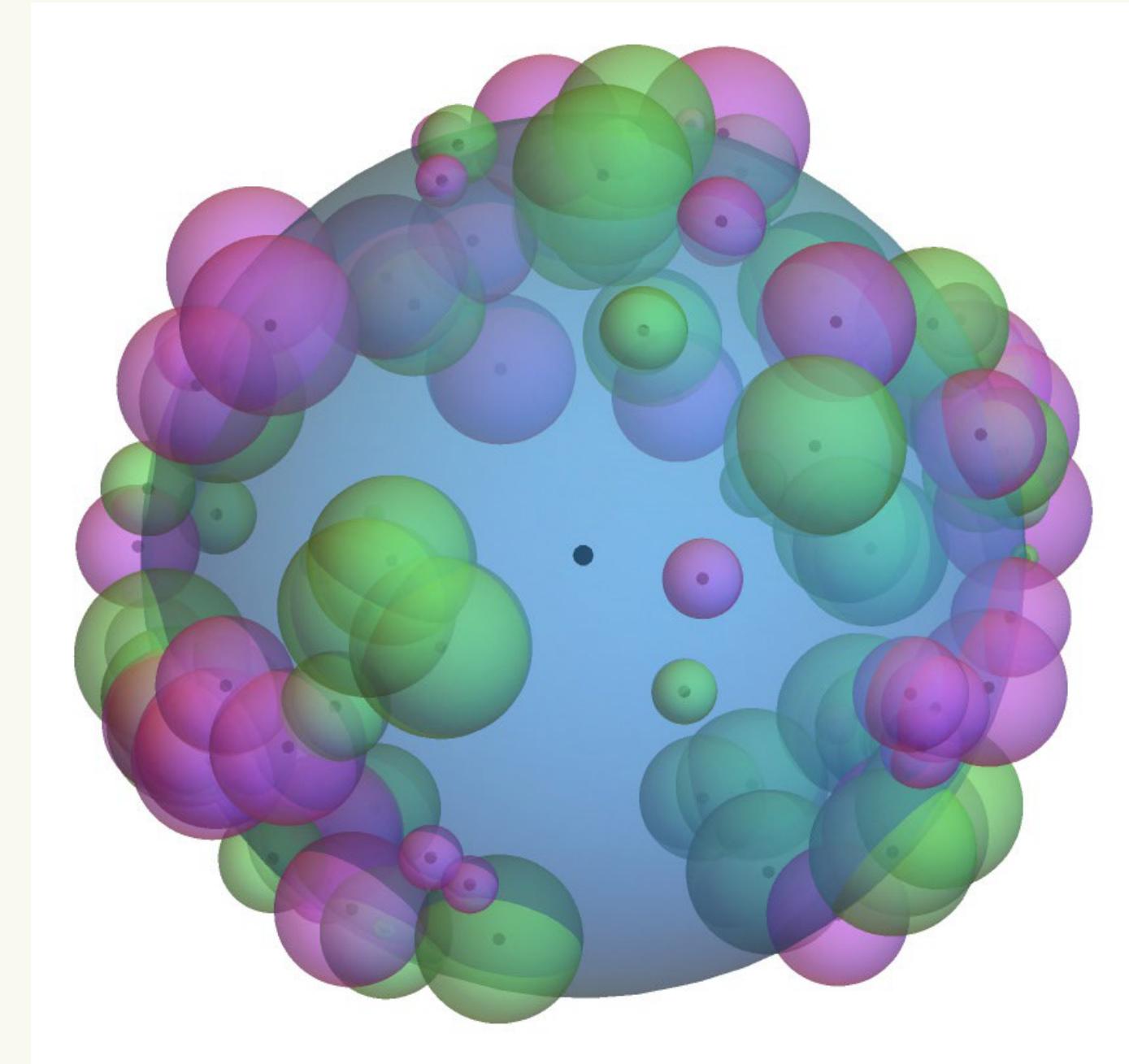
Parametric Geometries

Generalization to non-scalar parameters

Round objects can be expressed as center point plus multiples of attitude by **u**, where **u** has lower grade by 2

$$\mathbf{p}\left(\mathbf{u}\right) = \operatorname{cen}\left(\mathbf{x}\right) + \mathbf{u}^{*} \vee \operatorname{att}\left(\mathbf{x}\right)$$

Parametric Geometries

Dipole has scalar parameter

Circle has vector parameter

Sphere has bivector parameter

$$\mathbf{p}(\mathbf{u}) = \text{cen}(\mathbf{x}) + \mathbf{u}^* \vee \text{att}(\mathbf{x})$$

Parametric Dipole
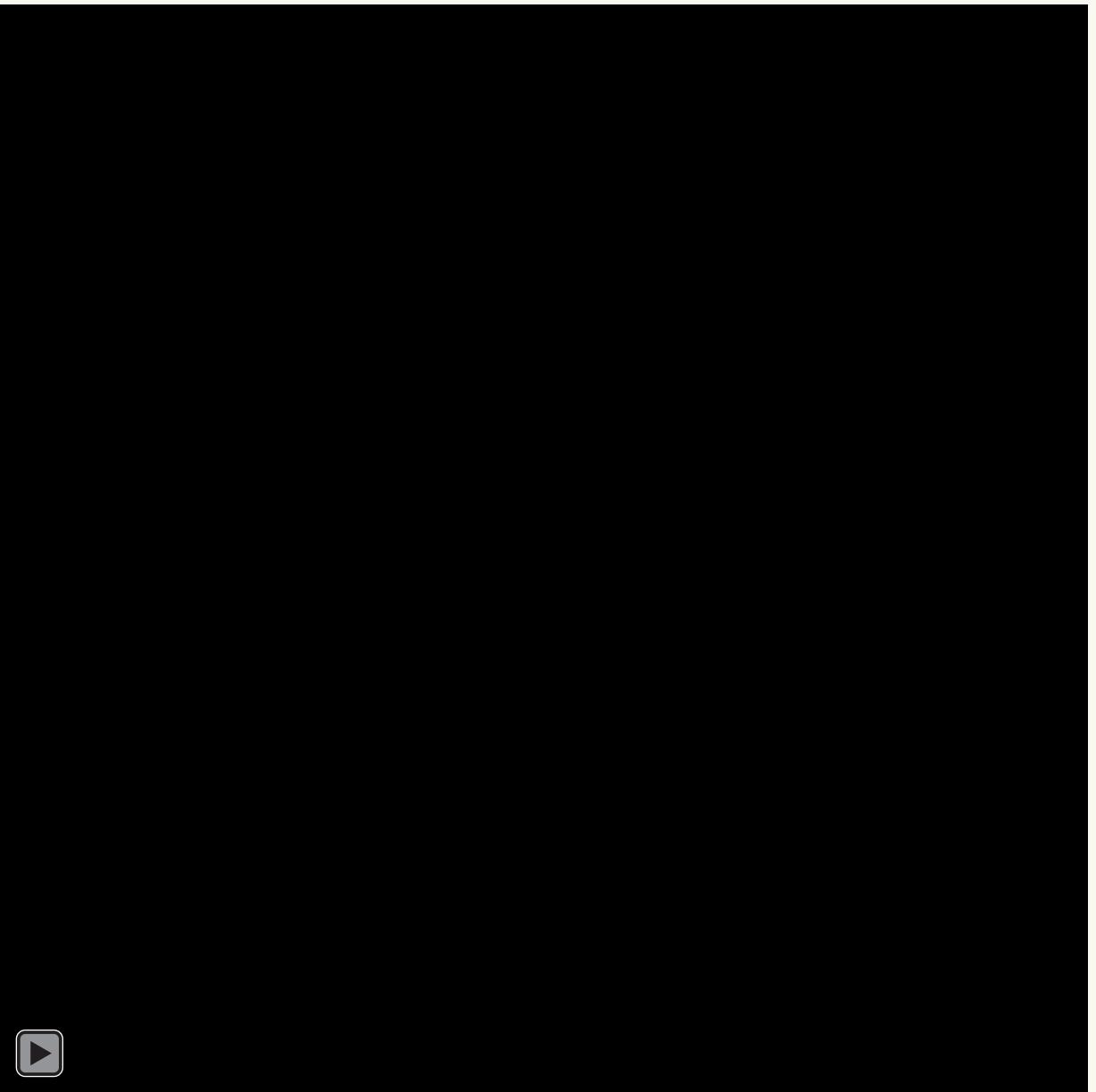
Sphere-Line Intersection

Calculate the points **a** and **b** where
a line *l* intersects a sphere having
center **c** and radius *r*.
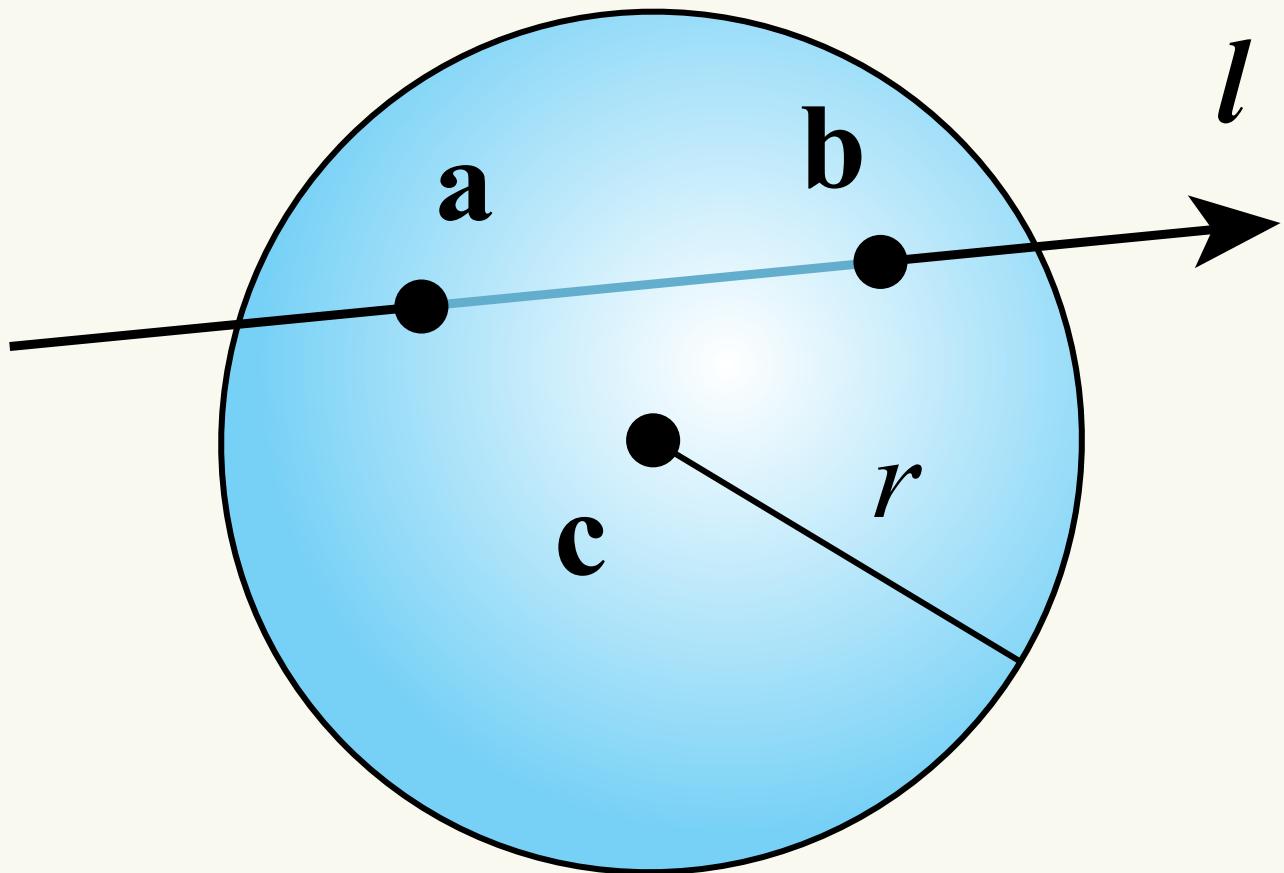
| Conventional Methods	Geometric Algebra
Let $l(t) = \mathbf{p} + t\mathbf{v}$ be a parametric line containing the point $\mathbf{p}$ and running parallel to the direction vector $\mathbf{v}$. Assume the direction is normalized so that $\|\mathbf{v}\| = 1$.	Let l be a flat line with direction l_v and moment l_m. Assume the line is weight normalized so that $\|l\|_\square = 1$. Assume the sphere is weight normalized so that $s_u = -1$.
Translate the center of the sphere to the origin. Translate the line by subtracting $\mathbf{c}$ from $\mathbf{p}$.	Translate the center of the sphere to the origin so it is given by $\mathbf{s} = -\mathbf{e}_{1234} + \frac{1}{2}r^2\mathbf{e}_{3215}$. Translate the line by subtracting $\mathbf{c} \times l_v$ from its moment l_m.
The goal is to solve the equation $(\mathbf{p} + t\mathbf{v})^2 = r^2$ for values of t and plug them into the line $\mathbf{p} + t\mathbf{v}$ to obtain the points $\mathbf{a}$ and $\mathbf{b}$.	The goal is to calculate endpoints of the dipole $\mathbf{d} = \mathbf{s} \vee l$, the meet of $\mathbf{s}$ and l. These are the points $\mathbf{a}$ and $\mathbf{b}$.
Expanding the quadratic equation, recognizing that $v^2 = 1$, and collecting terms in powers of t, we have $$t^2 + 2(\mathbf{p} \cdot \mathbf{v})t + p^2 - r^2 = 0$$	Applying $s_x = s_y = s_z = 0$, the dipole $\mathbf{d}$ is given by $$\mathbf{d} = -l_{vx}\mathbf{e}_{41} - l_{vy}\mathbf{e}_{42} - l_{vz}\mathbf{e}_{43} - l_{mx}\mathbf{e}_{23} - l_{my}\mathbf{e}_{31} - l_{mz}\mathbf{e}_{12}$$ $$+ \frac{1}{2}r^2(l_{vx}\mathbf{e}_{15} + l_{vy}\mathbf{e}_{25} + l_{vz}\mathbf{e}_{35})$$
The discriminant δ of the polynomial is given by $$\delta = (\mathbf{p} \cdot \mathbf{v})^2 - p^2 + r^2$$ If $\delta < 0$, then the line does not intersect the sphere.	Using $l_v^2 = 1$, the squared radius of the dipole is given by $$\|\mathbf{d}\|_\ominus^2 = r^2 - l_m^2$$ If $\|\mathbf{d}\|_\ominus^2 < 0$, then the line does not intersect the sphere.
The parameter values where the line intersects the sphere are given by $t = -\mathbf{p} \cdot \mathbf{v} \pm \sqrt{\delta}$. Plugging these into the line $l(t)$, the points $\mathbf{a}$ and $\mathbf{b}$ are obtained with $$\mathbf{a} = \mathbf{q} - \mathbf{v}\sqrt{\delta} \text{ and } \mathbf{b} = \mathbf{q} + \mathbf{v}\sqrt{\delta},$$ where $\mathbf{q} = \mathbf{p} - (\mathbf{p} \cdot \mathbf{v})\mathbf{v} + \mathbf{c}$ is the midpoint between them.	The center of the dipole is given by $$\text{cen}(\mathbf{d}) = (l_{vy}l_{mz} - l_{vz}l_{my})\mathbf{e}_1 + (l_{vz}l_{mx} - l_{vx}l_{mz})\mathbf{e}_2$$ $$+ (l_{vx}l_{my} - l_{vy}l_{mx})\mathbf{e}_3 + \mathbf{e}_4 + \frac{1}{2}r^2\mathbf{e}_5$$ The points $\mathbf{a}$ and $\mathbf{b}$ are obtained with $$\mathbf{a} = \mathbf{q} - l_v\|\mathbf{d}\|_\ominus \text{ and } \mathbf{b} = \mathbf{q} + l_v\|\mathbf{d}\|_\ominus,$$ where $\mathbf{q} = \text{cen}(\mathbf{d}) + \mathbf{c}$ is the midpoint between them.

GDC

Are Two Circles Linked?

Given two arbitrarily oriented circles,
how can we tell if they are linked?

Simply take antiwedge product to give round point

Radius of point is real if linked, imaginary otherwise

Are Two Circles Linked?

References

projectivegeometricalgebra.org

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